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Cover Figure: Typical paths for the deflator ψ, a universal consumption signal L, and the induced level of satisfaction Y^{C^n}, by courtesy of P. Bank and H. Föllmer

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Preface

This is the first volume of the Paris-Princeton Lectures in Financial Mathematics. The goal of this series is to publish cutting edge research in self-contained articles prepared by well known leaders in the field, or promising young researchers invited by the editors to contribute to a volume. Particular attention is paid to the quality of the exposition and we aim at articles that can serve as an introductory reference for research in the field.

The series is a result of frequent exchanges between researchers in finance and financial mathematics in Paris and Princeton. Many of us felt that the field would benefit from timely exposés of topics in which there is important progress. René Carmona, Erhan Cinlar, Ivar Ekeland, Elyes Jouini, José Scheinkman and Nizar Touzi will serve in the first editorial board of the Paris-Princeton Lectures in Financial Mathematics. Although many of the chapters in future volumes will involve lectures given in Paris or Princeton, we will also invite other contributions. Given the current nature of the collaboration between the two poles, we expect to produce a volume per year. Springer Verlag kindly offered to host this enterprise under the umbrella of the Lecture Notes in Mathematics series, and we are thankful to Catriona Byrne for her encouragement and her help in the initial stage of the initiative.

This first volume contains four chapters. The first one was written by Peter Bank and Hans Föllmer. It grew out of a seminar course at given at Princeton in 2002. It reviews a recent approach to optimal stopping theory which complements the traditional Snell envelop view. This approach is applied to utility maximization of a satisfaction index, American options, and multi-armed bandits.

The second chapter was written by Fabrice Baudoin. It grew out of a course given at CREST in November 2001. It contains an interesting, and very promising, extension of the theory of initial enlargement of filtration, which was the topic of his Ph.D. thesis. Initial enlargement of filtrations has been widely used in the treatment of asymetric information models in continuous-time finance. This classical view assumes the knowledge of some random variable in the almost sure sense, and it is well known that it leads to arbitrage at the final resolution time of uncertainty. Baudoin's chapter offers a self-contained review of the classical approach, and it gives a complete analysis of the case where the additional information is restricted to the distribution of a random variable.

The chapter contributed by Chris Rogers is based on a short course given during the *Montreal Financial Mathematics and Econometrics Conference* organized in June 2001 by CIRANO in Montreal. The aim of this event was to bring together leading experts and some of the most promising young researchers in both fields in order to enhance existing collaborations and set the stage for new ones. Roger's contribution gives an intuitive presentation of the duality approach to utility maximization problems in different contexts of market imperfections.

The last chapter is due to Mete Soner and Nizar Touzi. It also came out of seminar course taught at Princeton University in 2001. It provides an overview of the duality approach to the problem of super-replication of contingent claims under portfolio constraints. A particular emphasis is placed on the limitations of this approach, which in turn motivated the introduction of an original geometric dynamic programming principle on the initial formulation of the problem. This eventually allowed to avoid the passage from the dual formulation.

It is anticipated that the publication of this first volume will coincide with the *Blaise Pascal International Conference in Financial Modeling*, to be held in Paris (July 1-3, 2003). This is the closing event for the prestigious *Chaire Blaise Pascal* awarded to Jose Scheinkman for two years by the *Ecole Normale Supérieure de Paris*.

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Contents

American Options, Multi–armed Bandits, and Optimal Consumption Plans: A Unifying View

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Summary. In this survey, we show that various stochastic optimization problems arising in option theory, in dynamical allocation problems, and in the microeconomic theory of intertemporal consumption choice can all be reduced to the same problem of representing a given stochastic process in terms of running maxima of another process. We describe recent results of Bank and El Karoui (2002) on the general stochastic representation problem, derive results in closed form for Lévy processes and diffusions, present an algorithm for explicit computations, and discuss some applications.

Key words: American options, Gittins index, multi–armed bandits, optimal consumption plans, optimal stopping, representation theorem, universal exercise signal.
AMS 2000 subject classification. 60G07, 60G40, 60H25, 91B16, 91B28.

1 Introduction

At first sight, the optimization problems of exercising an American option, of allocating effort to several parallel projects, and of choosing an intertemporal consumption plan seem to be rather different in nature. It turns out, however, that they are all related to the same problem of representing a stochastic process in terms of running maxima of another process. This stochastic representation provides a new method for solving such problems, and it is also of intrinsic mathematical interest. In this survey, our purpose is to show how the representation problem appears in these different contexts, to explain and to illustrate its general solution, and to discuss some of its practical implications.

As a first case study, we consider the problem of choosing a consumption plan under a cost constraint which is specified in terms of a complete financial market

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model. Clearly, the solution depends on the agent's preferences on the space of consumption plans, described as optional random measures on the positive time axis. In the standard formulation of the corresponding optimization problem, one restricts attention to absolutely continuous measures admitting a rate of consumption, and the utility functional is a time–additive aggregate of utilities applied to consumption rates. However, as explained in [25], such time–additive utility functionals have serious conceptual deficiencies, both from an economic and from a mathematical point of view. As an alternative, Hindy, Huang and Kreps [25] propose a different class of utility functionals where utilities at different times depend on an index of satisfaction based on past consumption. The corresponding singular control problem raises new mathematical issues. Under Markovian assumptions, the problem can be analyzed using the Hamilton–Jacobi–Bellman approach; see [24] and [8]. In a general semimartingale setting, Bank and Riedel [6] develop a different approach. They reduce the optimization problem to the problem of representing a given process X in terms of running suprema of another process ξ:

$$X_t = \mathbb{E}\left[\int_{(t,+\infty]} f(s, \sup_{v\in[t,s)} \xi_v)\, \mu(ds) \,\middle|\, \mathcal{F}_t \right] \quad (t \in [0,+\infty))\,. \tag{1}$$

In the context of intertemporal consumption choice, the process X is specified in terms of the price deflator; the function f and the measure μ reflect the structure of the agent's preferences. The process ξ determines a minimal level of satisfaction, and the optimal consumption plan consists in consuming just enough to ensure that the induced index of satisfaction stays above this minimal level. In [6], the representation problem is solved explicitly under the assumption that randomness is modelled by a Lévy process.

In its general form, the stochastic representation problem (1) has a rich mathematical structure. It raises new questions even in the deterministic case, where it leads to a time–inhomogeneous notion of convex envelope as explained in [5]. In discrete time, existence and uniqueness of a solution easily follow by backwards induction. The stochastic representation problem in continuous time is more subtle. In a discussion of the first author with Nicole El Karoui at an Oberwolfach meeting, it became clear that it is closely related to the theory of Gittins indices in continuous time as developed by El Karoui and Karatzas in[17].

Gittins indices occur in the theory of multi–armed bandits. In such dynamic allocation problems, there is a a number of parallel projects, and each project generates a specific stochastic reward proportional to the effort spent on it. The aim is to allocate the available effort to the given projects so as to maximize the overall expected reward. The crucial idea of [23] consists in reducing this multi–dimensional optimization problem to a family of simpler benchmark problems. These problems yield a performance measure, now called the Gittins index, separately for each project, and an optimal allocation rule consists in allocating effort to those projects whose current Gittins index is maximal. [23] and [36] consider a discrete–time Markovian setting, [28] and [32] extend the analysis to diffusion models. El Karoui and Karatzas [17] develop a general martingale approach in continuous time. One of their results shows that Gittins indices can be viewed as solutions to a representation problem of the form (1). This connection turned out to be the key to the solution of the general representation problem in [5]. This representation result can be used as an alternative way to define Gittins indices, and it offers new methods for their computation.

As another case study, we consider American options. Recall that the holder of such an option has the right to exercise the option at any time up to a given deadline. Thus, the usual approach to option pricing and to the construction of replicating strategies has to be combined with an optimal stopping problem: Find a stopping time which maximizes the expected payoff. From the point of view of the buyer, the expectation is taken with respect to a given probabilistic model for the price fluctuation of the underlying. From the point of view of the seller and in the case of a complete financial market model, it involves the unique equivalent martingale measure. In both versions, the standard approach consists in identifying the optimal stopping times in terms of the Snell envelope of the given payoff process; see, e.g., [29]. Following [4], we are going to show that, alternatively, optimal stopping times can be obtained from a representation of the form (1) via a level crossing principle: A stopping time is optimal iff the solution ξ to the representation problem passes a certain threshold. As an application in Finance, we construct a universal exercise signal for American put options which yields optimal stopping rules simultaneously for all possible strikes. This part of the paper is inspired by a result in [18], as explained in Section 2.1.

The reduction of different stochastic optimization problems to the stochastic representation problem (1) is discussed in Section 2. The general solution is explained in Section 3, following [5]. In Section 4 we derive explicit solutions to the representation problem in homogeneous situations where randomness is generated by a Lévy process or by a one–dimensional diffusion. As a consequence, we obtain explicit solutions to the different optimization problems discussed before. For instance, this yields an alternative proof of a result by [33], [1], and [10] on optimal stopping rules for perpetual American puts in a Lévy model.

Closed–form solutions to stochastic optimization problems are typically available only under strong homogeneity assumptions. In practice, however, inhomogeneities are hard to avoid, as illustrated by an American put with finite deadline. In such cases, closed–form solutions cannot be expected. Instead, one has to take a more computational approach. In Section 5, we present an algorithm developed in [3] which explicitly solves the discrete–time version of the general representation problem (1). In the context of American options, for instance, this algorithm can be used to compute the universal exercise signal as illustrated in Figure 1.

Acknowledgement. We are obliged to Nicole El Karoui for introducing the first author to her joint results with Ioannis Karatzas on Gittins indices in continuous time; this provided the key to the general solution in [5] of the representation result discussed in this survey. We would also like to thank Christian Foltin for helping with the C++ implementation of the algorithm presented in Section 5.

Notation. *Throughout this paper we fix a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and a filtration* $(\mathcal{F}_t)_{t\in[0,+\infty]}$ *satisfying the usual conditions. By* $\mathcal{T}$ *we shall denote the set of all stopping times* $T \geq 0$*. Moreover, for a (possibly random) set* $A \subset [0,+\infty]$*,* $\mathcal{T}(A)$

will denote the class of all stopping times $T \in \mathcal{T}$ taking values in A almost surely. For instance, given a stopping time S, we shall make frequent use of $\mathcal{T}((S,+\infty])$ in order to denote the set of all stopping times $T \in \mathcal{T}$ such that $T(\omega) \in (S(\omega),+\infty]$ for almost every ω. For a given process $X = (X_t)$ we use the convention $X_{+\infty} = 0$ unless stated otherwise.

2 Reducing Optimization Problems to a Representation Problem

In this section we consider a variety of optimization problems in continuous time including optimal stopping problems arising in Mathematical Finance, a singular control problem from the microeconomic theory of intertemporal consumption choice, and the multi–armed bandit problem in Operations Research. We shall show how each of these different problems can be reduced to the same problem of representing a given stochastic process in terms of running suprema of another process.

2.1 American Options

An American option is a contingent claim which can be exercised by its holder at any time up to a given terminal time $\hat{T} \in (0,+\infty]$. It is described by a nonnegative, optional process $X = (X_t)_{t\in[0,\hat{T}]}$ which specifies the contingent payoff X_t if the option is exercised at time $t \in [0,\hat{T}]$.

A key example is the American put option on a stock which gives its holder the right to sell the stock at a price $k \geq 0$, the so–called strike price, which is specified in advance. The underlying financial market model is defined by a stock price process $P = (P_t)_{t\in[0,\hat{T}]}$ and an interest rate process $(r_t)_{t\in[0,\hat{T}]}$. For notational simplicity, we shall assume that interest rates are constant: $r_t \equiv r > 0$. The discounted payoff of the put option is then given by the process

$$X_t^k = e^{-rt}(k - P_t)^+ \quad (t \in [0,\hat{T}])\,.$$

Optimal Stopping via Snell Envelopes

The holder of an American put–option will try to maximize the expected proceeds by choosing a suitable exercise time. For a general optional process X, this amounts to the following optimal stopping problem:

$$\text{Maximize } \mathbb{E}X_T \text{ over all stopping times } T \in \mathcal{T}([0,\hat{T}])\,.$$

There is a huge literature on such optimal stopping problems, starting with [35]; see [16] for a thorough analysis in a general setting. The standard approach uses the theory of the *Snell envelope* defined as the unique supermartingale U such that

$$U_S = \operatorname*{ess\,sup}_{T\in\mathcal{T}([S,\hat{T}])} \mathbb{E}\left[X_T \,|\, \mathcal{F}_S\right]$$

for all stopping times $S \in \mathcal{T}([0,\hat{T}])$. Alternatively, the Snell envelope U can be characterized as the smallest supermartingale which dominates the payoff process X. With this concept at hand, the solution of the optimal stopping problem can be summarized as follows; see Théorème 2.43 in [16]:

Theorem 1. *Let X be a nonnegative optional process of class (D) which is upper–semicontinuous in expectation. Let U denote its Snell envelope and consider its Doob–Meyer decomposition $U = M - A$ into a uniformly integrable martingale M and a predictable increasing process A starting in $A_0 = 0$. Then*

$$\underline{T} \overset{\Delta}{=} \inf\{t \geq 0 \mid X_t = U_t\} \quad \textit{and} \quad \overline{T} \overset{\Delta}{=} \inf\{t \geq 0 \mid A_t > 0\} \tag{2}$$

are the smallest and the largest stopping times, respectively, which attain

$$\sup_{T \in \mathcal{T}([0,\hat{T}])} \mathbb{E} X_T \,.$$

In fact, a stopping time $T^* \in \mathcal{T}([0,\hat{T}])$ is optimal in this sense iff

$$\underline{T} \leq T^* \leq \overline{T} \quad \textit{and} \quad X_{T^*} = U_{T^*} \quad \mathbb{P}\textit{–a.s.} \tag{3}$$

Remark 1. 1. Recall that an optional process X is said to be *of class (D)* if $(X_T,\ T \in \mathcal{T})$ defines a uniformly integrable family of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$; see, e.g., [14]. Since we use the convention $X_{+\infty} \equiv 0$, an optional process X will be of class (D) iff

$$\sup_{T \in \mathcal{T}} \mathbb{E}|X_T| < +\infty \,,$$

and in this case the optimal stopping problem has a finite value.

2. As in [16], we call an optional process X of class (D) *upper–semicontinuous in expectation* if for any monotone sequence of stopping times T^n $(n = 1, 2, \ldots)$ converging to some $T \in \mathcal{T}$ almost surely, we have

$$\limsup_n \mathbb{E} X_{T^n} \leq \mathbb{E} X_T \,.$$

In the context of optimal stopping problems, upper–semicontinuity in expectation is a very natural assumption.

Applied to the American put option on P with strike $k > 0$, the theorem suggests that one should first compute the Snell envelope

$$U^k_S = \operatorname*{ess\,sup}_{T \in \mathcal{T}([S,\hat{T}])} \mathbb{E}\left[e^{-rT}(k - P_T)^+ \,\middle|\, \mathcal{F}_S\right] \quad (S \in \mathcal{T}([0,\hat{T}]))\,.$$

and then exercise the option, e.g., at time

$$\underline{T}^k = \inf\{t \geq 0 \mid U^k_t = e^{-rt}(k - P_t)^+\}\,.$$

For a fixed strike k, this settles the problem from the point of view of the option holder. From the point of view of the option seller, Karatzas [29] shows that the problem of pricing and hedging an American option in a complete financial market model amounts to the same optimal stopping problem, but in terms of the unique equivalent martingale measure $\mathbb{P}^*$ rather than the original measure $\mathbb{P}$. For a discussion of the incomplete case, see, e.g., [22].

A Level Crossing Principle for Optimal Stopping

In this section, we shall present an alternative approach to optimal stopping problems which is developed in [4], inspired by the discussion of American options in [18]. This approach is based on a representation of the underlying optional process X in terms of running suprema of another process ξ. The process ξ will take over the role of the Snell envelope, and it will allow us to characterize optimal stopping times by a *level crossing principle*.

Theorem 2. *Suppose that the optional process X admits a representation of the form*

$$X_T = \mathbb{E}\left[\int_{(T,+\infty]} \sup_{v \in [T,t)} \xi_v \, \mu(dt) \,\middle|\, \mathcal{F}_T \right] \quad (T \in \mathcal{T}) \tag{4}$$

for some nonnegative, optional random measure μ on $([0,+\infty], \mathcal{B}([0,+\infty]))$ and some progressively measurable process ξ with upper–right continuous paths such that

$$\sup_{v \in [T(\omega),t)} \xi_v(\omega) 1_{(T(\omega),+\infty]}(t) \in L^1(\mathbb{P}(d\omega) \otimes \mu(\omega, dt))$$

for all $T \in \mathcal{T}$.

Then the level passage times

$$\underline{T} \stackrel{\Delta}{=} \inf\{t \geq 0 \mid \xi_t \geq 0\} \quad \textit{and} \quad \overline{T} \stackrel{\Delta}{=} \inf\{t \geq 0 \mid \xi_t > 0\} \tag{5}$$

maximize the expected reward $\mathbb{E}X_T$ over all stopping times $T \in \mathcal{T}$.

If, in addition, μ has full support $\operatorname{supp} \mu = [0,+\infty]$ almost surely, then $T^* \in \mathcal{T}$ maximizes $\mathbb{E}X_T$ over $T \in \mathcal{T}$ iff

$$\underline{T} \leq T^* \leq \overline{T} \ \ \mathbb{P}\text{–a.s.} \quad \textit{and} \quad \sup_{v \in [0,T^*]} \xi_v = \xi_{T^*} \ \ \mathbb{P}\text{–a.s. on } \{T^* < +\infty\}\,. \tag{6}$$

In particular, $\underline{T}$ is the minimal and $\overline{T}$ is the maximal stopping time yielding an optimal expected reward.

Proof. Use (4) and the definition of $\overline{T}$ to obtain for any $T \in \mathcal{T}$ the estimates

$$\mathbb{E}X_T \leq \mathbb{E}\int_{(T,+\infty]} \sup_{v \in [0,t)} \xi_v \vee 0 \, \mu(dt) \leq \mathbb{E}\int_{(\overline{T},+\infty]} \sup_{v \in [0,t)} \xi_v \, \mu(dt)\,. \tag{7}$$

Choosing $T = \underline{T}$ or $T = \overline{T}$, we obtain equality in the first estimate since, for either choice, T is a level passage time for ξ so that

$$\sup_{v \in [0,t)} \xi_v = \sup_{v \in [T,t)} \xi_v \geq 0 \quad \text{for all} \quad t \in (T,+\infty]\,. \tag{8}$$

Since $T \leq \overline{T}$ in either case, we also have equality in the second estimate. Hence, both $T = \underline{T}$ and $T = \overline{T}$ attain the upper bound on $\mathbb{E}X_T$ $(T \in \mathcal{T})$ provided by these estimates and are therefore optimal.

It follows that a stopping time T^* is optimal iff equality holds true in both estimates occurring in (7). If μ has full support almost surely, it is easy to see that equality holds true in the second estimate iff $T^* \leq \overline{T}$ almost surely. Moreover, equality in the first estimate means exactly that (8) holds true almost surely. This condition, however, is equivalent to

$$\lim_{t \downarrow T^*} \sup_{v \in [0,t)} \xi_v = \limsup_{t \searrow T^*} \xi_t \geq 0 \;\; \mathbb{P}\text{–a.s. on } \{T^* < +\infty\}$$

which, by upper–right continuity of ξ, amounts to

$$\sup_{v \in [0,T^*]} \xi_v = \xi_{T^*} \geq 0 \;\; \mathbb{P}\text{–a.s. on } \{T^* < +\infty\}\,.$$

Equivalently:

$$T^* \geq \underline{T} \;\; \mathbb{P}\text{–a.s.} \quad \text{and} \quad \sup_{v \in [0,T^*]} \xi_v = \xi_{T^*} \geq 0 \;\; \mathbb{P}\text{–a.s. on } \{T^* < +\infty\}\,.$$

Thus, optimality of T^* is in fact equivalent to (6) if μ has full support almost surely.
□

Remark 2. 1. In Section 3, Theorem 6, we shall prove that an optional process $X = (X_t)_{t \in [0,+\infty]}$ of class (D) admits a representation of the form (4) if it is upper–semicontinuous in expectation. Moreover, Theorem 6 shows that we are free to choose an arbitrary measure μ from the class of all atomless, optional random measures on $[0,+\infty]$ with full support and finite expected total mass $\mathbb{E}\mu([0,+\infty]) < +\infty$. This observation will be useful in our discussion of American options in the next section.

2. The assumption that ξ is upper–right continuous, i.e., that

$$\xi_t = \limsup_{s \searrow t} \xi_s = \lim_{s \downarrow t} \sup_{v \in [t,s)} \xi_v \quad \text{for all} \quad t \in [0,+\infty) \quad \mathbb{P}\text{–a.s.},$$

can be made without loss of generality. Indeed, since a real function ξ and its upper–right continuous modification $\tilde{\xi}_t \stackrel{\Delta}{=} \limsup_{s \searrow t} \xi_s$ have the same supremum over sets of the form $[T,t)$, representation (4) is invariant under an upper–right continuous modification of the process ξ. The resulting process $\tilde{\xi}$ is again a progressively measurable process; see, e.g., from Théorème IV.90 of [13].

3. The *level crossing principle* established in Theorem 2 also holds if we start at a fixed stopping time $S \in \mathcal{T}$: A stopping time $T^*_S \in \mathcal{T}([S,+\infty])$ attains

$$\operatorname*{ess\,sup}_{T \in \mathcal{T}([S,+\infty])} \mathbb{E}\left[\, X_T \,|\, \mathcal{F}_S \right]$$

iff

$$\underline{T}_S \leq T^*_S \leq \overline{T}_S \;\; \mathbb{P}\text{–a.s.} \quad \text{and} \quad \sup_{v \in [S,T^*_S]} \xi_v = \xi_{T^*_S} \text{ on } \{T^*_S < +\infty\} \;\; \mathbb{P}\text{–a.s.}\,,$$

where $\underline{T}_S$ and $\overline{T}_S$ denote the level passage times

$$\underline{T}_S \triangleq \inf\{t \geq S \mid \xi_t \geq 0\} \quad \text{and} \quad \overline{T}_S \triangleq \inf\{t \geq S \mid \xi_t > 0\}\,.$$

This follows as in the proof of Theorem 2, using conditional expectations instead of ordinary ones.

The preceding theorem reduces the optimal stopping problem to a representation problem of the form (4) for optional processes. In order to see the relation to the Snell envelope U of X, consider the right continuous supermartingale V given by

$$V_t \triangleq \mathbb{E}\left[\int_{(t,\hat{T}]} \zeta_s\,\mu(ds)\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}\left[\int_{(0,\hat{T}]} \zeta_s\,\mu(ds)\,\middle|\,\mathcal{F}_t\right] - \int_{(0,t]} \zeta_s\,\mu(ds)$$

where

$$\zeta_s \triangleq \sup_{v\in[0,s)} \xi_v \vee 0 \quad (s \in [0,\hat{T}]).$$

Since $V \geq X$, the supermartingale V dominates the Snell envelope U of X. On the other hand,

$$V_t = \mathbb{E}\left[\int_{(\overline{T},\hat{T}]} \zeta_s\,\mu(ds)\,\middle|\,\mathcal{F}_t\right] = \mathbb{E}\,[X_{\overline{T}}\,|\,\mathcal{F}_t] \leq U_t \quad \text{on} \quad \{\overline{T} \geq t\}\,,$$

and so V coincides with U up to time $\overline{T}$. Is is easy to check that the stopping times $\underline{T}$ and $\overline{T}$ appearing in (2) and (5) are actually the same and that for any stopping T^* with $\underline{T} \leq T^* \leq \overline{T}$ a.s., the condition $U_{T^*} = X_{T^*}$ in (3) is equivalent to the condition $\sup_{v\in[0,T^*]} \xi_v = \xi_{T^*}$ in (6).

A representation of the form (4) can also be used to construct an alternative kind of envelope Y for the process X, as described in the following corollary. Part (iii) shows that Y can replace the Snell envelope of Theorem 1 as a reference process for characterizing optimal stopping times. Parts (i) and (ii) are taken from [5]. The process Y can also be viewed as a solution to a variant of Skorohod's obstacle problem; see Remark 4.

Corollary 1. *Let μ be a nonnegative optional random measure on $[0,+\infty]$ with full support* supp $\mu = [0,+\infty]$ *almost surely and consider an optional process X of class (D) with $X_{+\infty} = 0$ $\mathbb{P}$–a.s.*

1. *There exists at most one optional process Y of the form*

$$Y_T = \mathbb{E}\left[\int_{(T,+\infty]} \eta_t\,\mu(dt)\,\middle|\,\mathcal{F}_T\right] \quad (T \in \mathcal{T}) \tag{9}$$

for some adapted, left continuous, nondecreasing process $\eta \in L^1(\mathbb{P}\otimes\mu)$ such that Y dominates X, i.e.,

$$Y_T \geq X_T \ \ \mathbb{P}\text{–a.s.} \quad \text{for any} \quad T \in \mathcal{T},$$

and such that $Y_T = X_T$ $\mathbb{P}$–a.s. for any point of increase T of η.

2. *If X admits a representation of the form* (4)*, then such a process Y does in fact exist, and the associated increasing process η is uniquely determined up to $\mathbb{P}$–indistinguishability on $(0,+\infty]$ via*

$$\eta_t = \sup_{v\in[0,t)} \xi_v \quad (t \in (0,+\infty])$$

where ξ is the progressively measurable process occurring in (4).

3. *A stopping time $T^* \in \mathcal{T}$ maximizes $\mathbb{E}X_T$ over all $T \in \mathcal{T}$ iff*

$$\underline{T} \le T^* \le \overline{T} \quad \textit{and} \quad Y_{T^*} = X_{T^*} \quad \mathbb{P}\text{–a.s.}$$

where $\underline{T}$ and $\overline{T}$ are the level passage times

$$\underline{T} \overset{\Delta}{=} \inf\{t \in (0,+\infty] \mid \eta_t \ge 0\} \quad \textit{and} \quad \overline{T} \overset{\Delta}{=} \inf\{t \in (0,+\infty] \mid \eta_t > 0\}\,.$$

Remark 3. A stopping time $T \in \mathcal{T}$ is called a *point of increase* for a left–continuous increasing process η if, $\mathbb{P}$–a.s. on $\{0 < T < +\infty\}$, $\eta_T < \eta_t$ for any $t \in (T,+\infty]$.

Proof.

1. In order to prove uniqueness, assume $\zeta \in L^1(\mathbb{P}\otimes\mu)$ is another adapted, left continuous and non–decreasing process such that the corresponding optional process

$$Z_T = \mathbb{E}\left[\int_{(T,+\infty]} \zeta_t \,\mu(dt) \,\middle|\, \mathcal{F}_T\right] \quad (T \in \mathcal{T})$$

dominates X and such that $Z_T = X_T$ for any time of increase $T \in \mathcal{T}$ for ζ. For $\varepsilon > 0$, consider the stopping times

$$S^\varepsilon \overset{\Delta}{=} \inf\{t \ge 0 \mid \eta_t > \zeta_t + \varepsilon\}$$

and

$$T^\varepsilon \overset{\Delta}{=} \inf\{t \ge S^\varepsilon \mid \zeta_t > \eta_t\}\,.$$

By left continuity of ζ, we then have $T^\varepsilon > S^\varepsilon$ on $\{S^\varepsilon < +\infty\}$. Moreover, S^ε is a point of increase for η and by assumption on η we thus have

$$X_{S^\varepsilon} = Y_{S^\varepsilon} = \mathbb{E}\left[\int_{(S^\varepsilon,T^\varepsilon]} \eta_t \,\mu(dt) \,\middle|\, \mathcal{F}_{S^\varepsilon}\right] + \mathbb{E}\left[\int_{(T^\varepsilon,+\infty]} \eta_t \,\mu(dt) \,\middle|\, \mathcal{F}_{S^\varepsilon}\right].$$

By definition of T^ε, the first of these conditional expectations is strictly larger than $\mathbb{E}\left[\int_{(S^\varepsilon,T^\varepsilon]} \zeta_t \,\mu(dt) \,\middle|\, \mathcal{F}_{S^\varepsilon}\right]$ on $\{T^\varepsilon > S^\varepsilon\} \supset \{S^\varepsilon < +\infty\}$. The second conditional expectation equals $\mathbb{E}\left[Y_{T^\varepsilon} \,|\, \mathcal{F}_{S^\varepsilon}\right]$ by definition of Y, and is thus at least as large as $\mathbb{E}\left[X_{T^\varepsilon} \,|\, \mathcal{F}_{S^\varepsilon}\right]$ since Y dominates X by assumption. Hence, on $\{S^\varepsilon < +\infty\}$ we obtain the apparent contradiction that almost surely

$$X_{S^\varepsilon} > \mathbb{E}\left[\int_{(S^\varepsilon,T^\varepsilon]} \zeta_t\,\mu(dt)\,\middle|\,\mathcal{F}_{S^\varepsilon}\right] + \mathbb{E}\left[X_{T^\varepsilon}\,\middle|\,\mathcal{F}_{S^\varepsilon}\right]$$
$$= \mathbb{E}\left[\int_{(S^\varepsilon,T^\varepsilon]} \zeta_t\,\mu(dt)\,\middle|\,\mathcal{F}_{S^\varepsilon}\right] + \mathbb{E}\left[Z_{T^\varepsilon}\,\middle|\,\mathcal{F}_{S^\varepsilon}\right]$$
$$= Z_{S^\varepsilon} \geq X_{S^\varepsilon}$$

where for the first equality we used $Z_{T^\varepsilon} = X_{T^\varepsilon}$ a.s. This equation holds true trivially on $\{T^\varepsilon = +\infty\}$ as $X_{+\infty} = 0 = Z_{+\infty}$ by assumption, and also on $\{T^\varepsilon < +\infty\}$ since T^ε is a point of increase for ζ on this set. Clearly, the above contradiction can only be avoided if $\mathbb{P}[S^\varepsilon < +\infty] = 0$, i.e., if $\eta \leq \zeta + \varepsilon$ on $[0,+\infty)$ almost surely. Since ε was arbitrary, this entails $\eta \leq \zeta$ on $[0,+\infty)$ $\mathbb{P}$–a.s. Reversing the roles of η and ζ in the above argument yields the converse inequality, and this proves that $Y = Z$ as claimed.

2. By our integrability assumption on the progressively measurable process ξ which occurs in the representation (4), the process $\eta_t = \sup_{v\in[0,t)} \xi_v$ $(t \in (0,+\infty])$ is $\mathbb{P}\otimes\mu$–integrable and the associated process Y with (9) is of class (D). To verify that Y has the desired properties, it only remains to show that $Y_T = X_T$ for any point of increase $T \in \mathcal{T}$ of η. So assume that $\eta_T < \eta_t$ for any $t \in (T,+\infty]$, $\mathbb{P}$–almost surely. Recalling the definition of η, this entails for $t \downarrow T$ that

$$\sup_{v\in[0,T)} \xi_v = \eta_T \leq \eta_{T+} \leq \limsup_{t\searrow T} \xi_t = \xi_T \quad \mathbb{P}\text{–a.s.}$$

where the last equality follows by upper–right continuity of ξ. Hence, $\eta_t = \sup_{v\in[0,t)} \xi_v = \sup_{v\in[T,t)} \xi_v$ for any $t \in (T,+\infty]$ almost surely and so we have in fact

$$Y_T = \mathbb{E}\left[\int_{(T,+\infty]} \eta_t\,\mu(dt)\,\middle|\,\mathcal{F}_T\right] = \mathbb{E}\left[\int_{(T,+\infty]} \sup_{v\in[T,t)} \xi_v\,\mu(dt)\,\middle|\,\mathcal{F}_T\right] = X_T$$

where the last equality follows from representation (4).

3. Since the right continuous modification of η is an increasing, adapted process, we can easily represent Y as required by Theorem 2:

$$Y_T = \mathbb{E}\left[\int_{(T,+\infty]} \sup_{v\in[T,t)} \eta_{v+}\,\mu(dt)\,\middle|\,\mathcal{F}_T\right] \quad (T \in \mathcal{T}).$$

Hence, the stopping times maximizing $\mathbb{E}Y_T$ over $T \in \mathcal{T}$ are exactly those stopping times T^* such that

$$\underline{T} \leq T^* \leq \overline{T} \;\; \mathbb{P}\text{–a.s.} \quad \text{and} \quad \sup_{v\in[0,T^*]} \eta_{v+} = \eta_{T^*+} \;\mathbb{P}\text{–a.s. on } \{T^* < +\infty\} \tag{10}$$

where

$$\underline{T} \stackrel{\Delta}{=} \inf\{t \in (0,+\infty] \mid \eta_{t+} \geq 0\} = \inf\{t \in (0,+\infty] \mid \eta_t \geq 0\}$$

and

$$\overline{T} \stackrel{\Delta}{=} \inf\{t \in (0,+\infty] \mid \eta_{t+} > 0\} = \inf\{t \in (0,+\infty] \mid \eta_t > 0\}\,.$$

By monotonicity of η, the second condition in (10) is actually redundant, and so a stopping time T^* is optimal for Y iff

$$\underline{T} \leq T^* \leq \overline{T} \ \ \mathbb{P}\text{–a.s.}$$

In particular, both $\underline{T}$ and $\overline{T}$ are optimal stopping times for Y. In addition, $\overline{T}$ is a time of increase for η. Thus, $X_{\overline{T}} = Y_{\overline{T}}$ $\mathbb{P}$–a.s. and

$$\max_{T\in\mathcal{T}} \mathbb{E}X_T \geq \mathbb{E}X_{\overline{T}} = \mathbb{E}Y_{\overline{T}} = \max_{T\in\mathcal{T}} \mathbb{E}Y_T\,.$$

But since $Y \geq X$ by assumption, we have in fact equality everywhere in the above expression, and so the values of the optimal stopping problems for X and Y coincide, and we obtain that any optimal stopping time T^* for X must satisfy $X_{T^*} = Y_{T^*}$ and it must also be an optimal stopping time for Y, i.e., satisfy $\underline{T} \leq T^* \leq \overline{T}$ almost surely. Conversely, an optimal stopping time T^* for Y which in addition satisfies $X_{T^*} = Y_{T^*}$ almost surely will also be optimal for X.
Let us finally prove that $\underline{T}$ is also an optimal stopping time for X. Since $\underline{T}$ is known to be optimal for Y it suffices by the above criterion to verify that $X_{\underline{T}} = \check{X}_{\underline{T}}$ almost surely. By definition of Y this identity holds true trivially on the set where η crosses the zero level by a jump at time $\underline{T}$, since then $\underline{T}$ is obviously a point of increase for η. To prove this identity also on the complementary set, consider the increasing sequence of stopping times

$$T^n \stackrel{\Delta}{=} \inf\{t \in [0,\underline{T}) \mid \eta_t > -1/n\} \quad (n = 1,2,\ldots)\,.$$

By definition, each T^n is a time of increase for η, and thus $X_{T^n} = Y_{T^n}$ holds true almost surely by the properties of Y. Moreover, the stopping times T^n increase to the restriction $\underline{T}'$ of $\underline{T}$ to the set where η continuously approaches its zero level:

$$T^n \to \underline{T}' = \begin{cases} \underline{T} & \text{on } \{\eta_{\underline{T}-} = 0\} \\ +\infty & \text{on } \{\eta_{\underline{T}-} < 0\} \end{cases}$$

Indeed, on $\{\underline{T}' < +\infty\}$, the stopping times T^n converge to $\underline{T}'$ strictly from below. It follows that

$$\begin{aligned} \mathbb{E}X_{T^n} = \mathbb{E}Y_{T^n} &= \mathbb{E}\int_{(T^n,+\infty]} \eta_t\,\mu(dt) \\ &\to \mathbb{E}\left[\int_{[\underline{T}',+\infty]} \eta_t\,\mu(dt)\ ;\ \underline{T}' < +\infty\right] = \mathbb{E}Y_{\underline{T}'}\,, \end{aligned}$$

where the last identity holds true because $\eta_{\underline{T}'} = 0$ on $\{\underline{T}' < +\infty\}$.
Since Y dominates X the right side of the above expression is $\geq \mathbb{E}X_{\underline{T}'}$. On the other hand, in the limit $n \uparrow +\infty$, its left side is not larger than $\mathbb{E}X_{\underline{T}'}$ since X is upper semicontinuous in expectation. Hence, we must have $\mathbb{E}Y_{\underline{T}'} = \mathbb{E}X_{\underline{T}'}$ which implies that in fact $Y_{\underline{T}'} = X_{\underline{T}'}$ almost surely, as we wanted to show. □

Remark 4. Parts (i) and (ii) of the above theorem can be seen as a uniqueness and existence result for a variant of Skorohod's obstacle problem, if the optional process X is viewed as a randomly fluctuating obstacle on the real line. With this interpretation, we can consider the set of all class (D) processes Y which never fall below the obstacle X and which follow a backward semimartingale dynamics of the form

$$dY_t = -\eta_t \, d\mu((0,t]) + dM_t \quad \text{and} \quad Y_{+\infty} = 0$$

for some uniformly integrable martingale M and for some adapted, left continuous, and non–decreasing process $\eta \in L^1(\mathbb{P} \otimes \mu)$. Rewriting the above dynamics in integral form and taking conditional expectations, we see that any such Y takes the form

$$Y_T = \mathbb{E}\left[\int_{(T,+\infty]} \eta_t \, \mu(dt) \,\middle|\, \mathcal{F}_T \right] \quad (T \in \mathcal{T}) .$$

Clearly, there will be many non–decreasing processes η which control the corresponding process Y in such a way that it never falls below the obstacle X. However, one could ask whether there is any such process η which only increases when necessary, i.e., when its associated process Y actually hits the obstacle X, and whether such a minimal process η is uniquely determined. The results of [5] as stated in parts (i) and (ii) of Corollary 1 give affirmative answers to both questions under general conditions.

Universal Exercise Signals for American Options

In the first part of the present section, we have seen how the optimal stopping problem for American options can be solved by using Snell envelopes. In particular, an American put option with strike k is optimally exercised, for instance, at time

$$T^k \triangleq \inf\{t \in [0,\hat{T}] \mid U^k_t = e^{-rt}(k - P_t)^+\} ,$$

where the process $(U^k_t)_t$ is defined as the Snell envelope of the discounted payoff process $(e^{-rt}(k-P_t)^+)_{t\in[0,\hat{T}]}$. Clearly, this construction of the optimal exercise rule is specific for the strike k considered. In practice, however, American put options are traded for a whole variety of different strike prices, and computing all relevant Snell envelopes may turn into a tedious task. Thus, it would be convenient to have a single reference process which allows one to determine optimal exercise times simultaneously for any possible strike k. In fact, it is possible to construct such a universal signal using the stochastic representation approach to optimal stopping developed in the preceding section:

Theorem 3. *Assume that the discounted value process $(e^{-rt}P_t)_{t\in[0,\hat{T}]}$ is an optional process of class (D) which is lower–semicontinuous in expectation.*

Then this process admits a unique representation

$$e^{-rT} P_T = \mathbb{E}\left[\int_{(T,\hat{T}]} re^{-rt} \inf_{v\in[T,t)} K_v \, dt + e^{-r\hat{T}} \inf_{v\in[T,\hat{T}]} K_v \,\middle|\, \mathcal{F}_T \right] \tag{11}$$

for $T \in \mathcal{T}([0,\hat{T}])$, and for some progressively measurable process $K = (K_t)_{t\in[0,\hat{T}]}$ with lower–right continuous paths such that

$$re^{-rt} \inf_{v\in[T,t)} K_v 1_{(T,\hat{T}]}(t) \in L^1(\mathbb{P}\otimes dt) \quad \text{and} \quad e^{-r\hat{T}} \inf_{v\in[T,\hat{T}]} K_v \in L^1(\mathbb{P})$$

for all $T \in \mathcal{T}([0,\hat{T}])$.

The process K provides a universal exercise signal for all American put options on the underlying process P in the sense that for any strike $k \geq 0$ the level passage times

$$\underline{T}^k \stackrel{\Delta}{=} \inf\{t \in [0,\hat{T}] \mid K_t \leq k\} \quad \text{and} \quad \overline{T}^k \stackrel{\Delta}{=} \inf\{t \in [0,\hat{T}] \mid K_t < k\}$$

provide the smallest and the largest solution, respectively, of the optimal stopping problem

$$\max_{T\in\mathcal{T}([0,\hat{T}]\cup\{+\infty\})} \mathbb{E}\left[e^{-rT}(k-P_T)\,;\,T\leq\hat{T}\right].$$

In fact, a stopping time $T^k \in \mathcal{T}([0,\hat{T}]\cup\{+\infty\})$ is optimal in this sense iff

$$\underline{T}^k \leq T^k \leq \overline{T}^k \quad \mathbb{P}\text{–a.s.} \quad \text{and} \quad \inf_{v\in[0,T^k]} K_v = K_{T^k} \ \mathbb{P}\text{–a.s. on } \{T^k \leq \hat{T}\}. \quad (12)$$

Remark 5. The preceding theorem is inspired by the results of El Karoui and Karatzas [18]. Their equation (1.4) states the following representation for the early exercise premium of an American put:

$$\begin{aligned}
&\operatorname*{ess\,sup}_{T\in\mathcal{T}([S,\hat{T}])} \mathbb{E}\left[e^{-r(T-S)}(k-P_T)^+ \,\middle|\, \mathcal{F}_S\right] - \mathbb{E}\left[e^{-r(\hat{T}-S)}(k-P_{\hat{T}})^+ \,\middle|\, \mathcal{F}_S\right] \\
&\quad = \mathbb{E}\Bigg[\int_{(S,T]} re^{-r(t-S)}\left(k - \inf_{v\in[S,t)} K_v\right)^+ dt \\
&\qquad\qquad + e^{-r(\hat{T}-S)}\left(k\wedge P_{\hat{T}} - \inf_{v\in[S,\hat{T})} K_v\right)^+ \,\Bigg|\, \mathcal{F}_S\Bigg].
\end{aligned}$$

This representation involves the same process K as considered in our Theorem 3. In fact, their formula (5.4), which in our notation reads

$$\begin{aligned}
&\lim_{k\uparrow+\infty}\left\{k - \operatorname*{ess\,sup}_{T\in\mathcal{T}([S,\hat{T}])} \mathbb{E}\left[e^{-r(T-S)}(k-P_T)^+ \,\middle|\, \mathcal{F}_S\right]\right\} \\
&\quad = \mathbb{E}\left[\int_{(T,\hat{T}]} re^{-r(t-S)} \inf_{v\in[T,t)} K_v\,dt + e^{-r(\hat{T}-S)} \inf_{v\in[T,\hat{T}]} K_v \,\middle|\, \mathcal{F}_S\right],
\end{aligned}$$

turns out to be identical with our equation (11) after noting that the limit on the left side coincides with the value of the underlying:

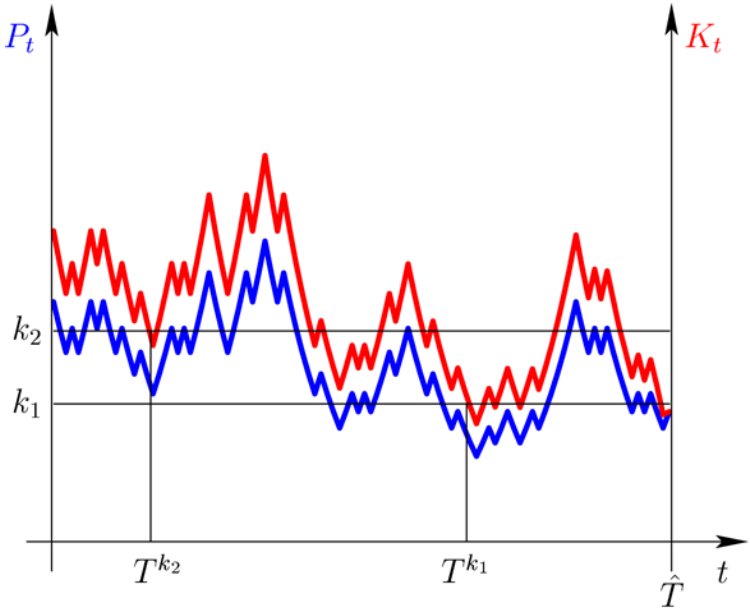

Fig. 1. Universal exercise signal K (red or light gray line) for an underlying P (blue or dark line), and optimal stopping times T^{k_1}, T^{k_2} for two different strikes $k_1 < k_2$ (black lines).

$$\lim_{k \uparrow +\infty} \left\{ k - \operatorname*{ess\,sup}_{T \in \mathcal{T}([S,\hat{T}])} \mathbb{E}\left[e^{-r(T-S)}(k - P_T)^+ \,\middle|\, \mathcal{F}_S \right] \right\} = P_S$$

$\mathbb{P}$–a.s. for all $S \in \mathcal{T}([0,\hat{T}])$. While we use the representation property (11) in order to define the process K, El Karoui and Karatzas introduce this process by a Gittins index principle: Their equation (1.3), which in our notation reads

$$K_S = \inf \left\{ k > 0 \mid \operatorname*{ess\,sup}_{T \in \mathcal{T}([S,\hat{T}])} \mathbb{E}\left[e^{-r(T-S)}(k - P_T)^+ \,\middle|\, \mathcal{F}_S \right] = k - P_S \right\},$$

with $S \in \mathcal{T}([0,\hat{T}]$, defines K_S as the minimal strike for which the corresponding American put is optimally exercised immediately at time S. Thus, the process K is specified in terms of Snell envelopes. In contrast, *our* approach defines K directly as the solution to the representation problem (11), and it emphasizes the role of K as a universal exercise signal. In homogeneous models, it is often possible to solve the representation problem directly, without first solving some optimization problem. This shortcut will be illustrated in Section 4 where we shall derive some explicit solutions.

Proof.

1. Existence of a representation for the discounted value process $(e^{-rt}P_t)_{t\in[0,\hat{T}]}$ as in (11) follows from a general representation theorem which will be proved in the next section; confer Corollary 3.
2. For any strike $k \geq 0$, let us consider the optional payoff process X^k defined by

$$X_t^k \stackrel{\Delta}{=} e^{-rt}(k - P_{t\wedge\hat{T}}) \quad (t \in [0, +\infty]) \,.$$

We claim that the stopping times T^k maximizing $\mathbb{E}X_T^k$ over $T \in \mathcal{T}$ are exactly those stopping times which maximize $\mathbb{E}\left[e^{-rT}(k - P_T)\,;\, T \leq \hat{T}\right]$ over $T \in \mathcal{T}([0,\hat{T}] \cup \{+\infty\})$. In fact, a stopping time $T^k \in \mathcal{T}$ maximizing $\mathbb{E}X_T^k$ will actually take values in $[0,\hat{T}] \cup \{+\infty\}$ almost surely because interest rates r are strictly positive by assumption. Hence, we have

$$\begin{aligned}\max_{T\in\mathcal{T}} \mathbb{E}X_T^k = \mathbb{E}X_{T^k}^k &= \mathbb{E}\left[e^{-rT^k}(k - P_{T^k})\,;\, T^k \leq \hat{T}\right] \\ &\leq \max_{T\in\mathcal{T}([0,\hat{T}]\cup\{+\infty\})} \mathbb{E}\left[e^{-rT}(k - P_T)\,;\, T \leq \hat{T}\right] .\end{aligned}$$

On the other hand, we have

$$\mathbb{E}\left[e^{-rT}(k - P_T)\,;\, T \leq \hat{T}\right] = \mathbb{E}X_T^k$$

for any $T \in \mathcal{T}([0,\hat{T}] \cup \{+\infty\})$, again by strict positivity of interest rates. As a consequence, the last $\max$ coincides with the first $\max$ and both lead to the same set of maximizers.
3. We wish to apply Theorem 2 in order to solve the optimal stopping problem for X^k $(k \geq 0)$ as defined in step (ii) of the present proof. To this end, let us construct a representation

$$X_T^k = \mathbb{E}\left[\int_{(T,+\infty]} \sup_{v\in[T,t)} \xi_v^k \,\mu(dt) \,\middle|\, \mathcal{F}_T\right] \quad (T \in \mathcal{T})$$

as required by this theorem. In fact, let

$$\xi_t^k \stackrel{\Delta}{=} k - K_{t\wedge\hat{T}} \quad (t \in [0, +\infty))$$

and put $\mu(dt) \stackrel{\Delta}{=} re^{-rt}\,dt$. Then ξ^k is obviously a progressively measurable process with upper–right continuous paths and we have for $T \in \mathcal{T}$:

$$
\begin{aligned}
&\mathbb{E}\left[\int_{(T,+\infty]} \sup_{v\in[T,t)} \xi^k_v\,\mu(dt)\,\middle|\,\mathcal{F}_T\right]\\
&= \mathbb{E}\left[\int_{(T,+\infty]} re^{-rt}(k - \inf_{v\in[T,t)} K_{v\wedge\hat{T}})\,dt\,\middle|\,\mathcal{F}_T\right]\\
&= e^{-rT}k - \mathbb{E}\left[\int_{(T\wedge\hat{T},\hat{T}]} re^{-rt} \inf_{v\in[T\wedge\hat{T},t)} K_v dt\right.\\
&\qquad\qquad \left. + \int_{(T\vee\hat{T},+\infty]} re^{-rt} \inf_{v\in[T\wedge\hat{T},\hat{T}]} K_v\,dt\,\middle|\,\mathcal{F}_T\right]\\
&= e^{-rT}k - \mathbb{E}\left[\int_{(T\wedge\hat{T},\hat{T}]} re^{-rt} \inf_{v\in[T\wedge\hat{T},t)} K_v + e^{-rT\vee\hat{T}} \inf_{v\in[T\wedge\hat{T},\hat{T}]} K_v\,\middle|\,\mathcal{F}_T\right]\\
&= e^{-rT}(k - P_{T\wedge\hat{T}})\,.
\end{aligned}
$$

Here, the last identity holds true on $\{T \leq \hat{T}\}$ because of the representation property (11) of K, and also on the complementary event $\{T > \hat{T}\}$, since on this set $\inf_{v\in[T\wedge\hat{T},\hat{T}]} K_v = K_{\hat{T}} = P_{\hat{T}}$, again by (11).

4. Applying Theorem 2 to $X = X^k$, we obtain that $T^k \in \mathcal{T}$ maximizes $\mathbb{E}X^k_T$ over all $T \in \mathcal{T}$ iff

$$
\underline{T}^k \leq T^k \leq \overline{T}^k \quad \mathbb{P}\text{–a.s.} \quad \text{and} \quad \sup_{v\in[0,T^k]} \xi^k_v = \xi^k_{T^k} \ \mathbb{P}\text{–a.s. on } \{T^k < +\infty\}\,,
$$

where $\underline{T}^k \triangleq \inf\{t \geq 0 \mid \xi^k_t \geq 0$ and $\overline{T}^k = \inf\{t \geq 0 \mid \xi^k_t > 0\}$. Recalling the definition of ξ^k and that $\{T^k < +\infty\} = \{T^k \leq \hat{T}\}$ for any optimal stopping time for X^k by (ii), we see that this condition is actually equivalent to the criterion in (12). □

Let us now apply Theorem 3 to the usual put option profile $(e^{-rt}(k-P)^+)_{t\in[0,\hat{T}]}$.

Corollary 2. *The universal exercise signal $K = (K_t)_{t\geq 0}$ characterized by* (11) *satisfies $K_T \geq P_T$ for all $T \in \mathcal{T}([0,\hat{T}])$ almost surely. In particular, the restriction $T^k \wedge \hat{T}$ of any optimal stopping time T^k as characterized in Theorem 3 also maximizes $\mathbb{E}e^{-rT}(k - P_T)^+$ among all stopping times $T \in \mathcal{T}([0,\hat{T}])$.*

Proof. For any $T \in \mathcal{T}([0,\hat{T}])$, the representation (11) implies

$$
\begin{aligned}
e^{-rT}P_T &= \mathbb{E}\left[\int_{(T,\hat{T}]} re^{-rt} \inf_{v\in[T,t)} K_v\,dt + e^{-r\hat{T}} \inf_{v\in[T,\hat{T}]} K_v\,\middle|\,\mathcal{F}_T\right]\\
&\leq \mathbb{E}\left[\int_{(T,\hat{T}]} re^{-rt}K_T\,dt + e^{-r\hat{T}}K_T\,\middle|\,\mathcal{F}_T\right]\\
&= e^{-rT}K_T
\end{aligned}
$$

almost surely. In particular, $P_{T^k} \leq K_{T^k} \leq k$ almost surely on $\{T^k \leq \hat{T}\}$ for any optimal stopping time T^k as in Theorem 3. Thus,

$$\mathbb{E}\left[e^{-rT^k}(k - P_{T^k})\ ;\ T^k \leq \hat{T}\right] = \mathbb{E}\left[e^{-rT^k \wedge T}(k - P_{T^k \wedge \hat{T}})^+\right]$$

and so $T^k \wedge \hat{T}$ maximizes $\mathbb{E}e^{-rT}(k - P_T)^+$ over $T \in \mathcal{T}([0, \hat{T}])$. □

Using the same arguments as in the proof of Theorem 4, we can also construct universal exercise signals for American call options:

Theorem 4. *Assume the discounted value process* $(e^{-rt}P_t)_{t\in[0,\hat{T}]}$ *is an optional process of class (D) which is upper–semicontinuous in expectation. Then this process admits a unique representation*

$$e^{-rT}P_T = \mathbb{E}\left[\int_{(T,\hat{T}]} re^{-rt} \sup_{v\in[T,t)} K_v\, dt + e^{-r\hat{T}} \sup_{v\in[T,\hat{T}]} K_v \,\middle|\, \mathcal{F}_T\right] \tag{13}$$

for $T \in \mathcal{T}([0, \hat{T}])$*, and for some progressively measurable process* K *with upper–right continuous paths and*

$$re^{-rt} \sup_{v\in[T,t)} K_s 1_{(T,\hat{T}]}(t) \in L^1(\mathbb{P} \otimes dt) \quad \textit{and} \quad e^{-r\hat{T}} \sup_{v\in[T,\hat{T}]} K_v \in L^1(\mathbb{P})$$

for all $T \in \mathcal{T}([0, \hat{T}])$.

This process K *provides a universal exercise signal for all American call options with underlying* P *in the sense that for any strike* $k \geq 0$ *the level passage times*

$$\underline{T}^k \triangleq \inf\{t \in [0, \hat{T}] \mid K_t \geq k\} \quad \textit{and} \quad \overline{T}^k \triangleq \inf\{t \in [0, \hat{T}] \mid K_t > k\}$$

provide the smallest and the largest solution, respectively, of the optimal stopping problem

$$\max_{T\in\mathcal{T}([0,\hat{T}]\cup\{+\infty\})} \mathbb{E}\left[e^{-rT}(P_T - k)\ ;\ T \leq \hat{T}\right].$$

In fact, a stopping time T^k *is optimal in this sense iff*

$$\underline{T}^k \leq T^k \leq \overline{T}^k \ \ \mathbb{P}\text{–a.s.} \quad \textit{and} \quad \sup_{v\in[0,T^k]} K_v = K_{T^k}\ \mathbb{P}\text{–a.s. on } \{T^k < +\infty\}.$$

The preceding theorem solves the optimal stopping problem of American calls under a general probability measure $\mathbb{P}$. For example, $\mathbb{P}$ could specify the probabilistic model used by the buyer of the option. From the point of view of the option seller and in the context of a complete financial market model, however, the problem should be formulated in terms of the equivalent martingale $\mathbb{P}^*$. In this case, the payoff process of the call option is a submartingale, and the optimal stopping problem is clearly solved by the simple rule: "Always stop at the terminal time $\hat{T}$". In the preceding theorem, this is reflected by the fact that the process K takes the simple form $K_t = 0$ for $t \in [0, \hat{T})$ and $K_{\hat{T}} = P_{\hat{T}}$.

Remark 6. The results of this section also apply when interest rates $r = (r_t)_{0 \le t \le \hat{T}}$ follow a progressively measurable process, provided this process is integrable and *strictly* positive. For instance, the representation (11) then takes the form

$$e^{-\int_0^T r_s\,ds} P_T = \mathbb{E}\left[\int_{(T,\hat{T}]} r_t e^{-\int_0^t r_s\,ds} \inf_{v \in [T,t)} K_v\,dt + e^{-\int_0^{\hat{T}} r_s\,ds} \inf_{v \in [T,\hat{T}]} K_v \,\middle|\, \mathcal{F}_T\right]$$

for $T \in \mathcal{T}([0,\hat{T}])$.

2.2 Optimal Consumption Plans

In this section, we discuss a singular control problem arising in the microeconomic theory of intertemporal consumption choice. We shall show how this problem can be reduced to a stochastic representation problem of the same type as in the previous section.

Consider an economic agent who makes a choice among different consumption plans. A consumption pattern is described as a positive measure on the time axis $[0,+\infty)$ or, in a cumulative way, by the corresponding distribution function. Thus, a consumption plan which is contingent on scenarios is specified by an element in the set

$$\mathcal{C} \stackrel{\Delta}{=} \{C \ge 0 \mid C \text{ is a right continuous, increasing and adapted process}\}.$$

Given some initial wealth $w > 0$, the agent's budget set is of the form

$$\mathcal{C}(w) \stackrel{\Delta}{=} \left\{C \in \mathcal{C} \,\middle|\, \mathbb{E}\int_{[0,+\infty)} \psi_t\,dC_t \le w\right\} \tag{14}$$

where $\psi = (\psi_t)_{t \in [0,+\infty)} > 0$ is a given optional price deflator.

Remark 7. Consider a financial market model specified by an $\mathbb{R}^d$–valued semimartingale $(P_t)_{t \in [0,+\infty)}$ of asset prices and an optional process $(r_t)_{t \in [0,+\infty)}$ of interest rates. Absence of arbitrage opportunities can be guaranteed by the existence of an equivalent local martingale measures $\mathbb{P}^* \approx \mathbb{P}$; cf. [12]. An initial capital V_0 is sufficient to implement a given consumption plan $C \in \mathcal{C}$ if there is a trading strategy, given by a d–dimensional predictable process $(\theta_t)_{t \in [0,+\infty)}$, such that the resulting wealth process

$$V_t = V_0 + \int_0^t \theta_s\,dP_s + \int_0^t (V_s - \theta_s P_s) r_s\,ds - C_t \quad (t \in [0,+\infty))$$

remains nonnegative. Thus, the cost of implementing the consumption plan C should be defined as the smallest such value V_0. Dually, this cost can be computed as

$$\sup_{\mathbb{P}^* \in \mathcal{P}^*} \mathbb{E}^* \int_0^{+\infty} e^{-\int_0^t r_s\,ds}\,dC_s\,,$$

where $\mathcal{P}^*$ denotes the class of all equivalent local martingale measures; this follows from a theorem on optional decompositions which was proved in increasing generality by [20], [31], and [21]. In the case of a complete financial market model, the equivalent martingale measure $\mathbb{P}^*$ is unique, and the cost takes the form appearing in (14), with

$$\psi_t \stackrel{\Delta}{=} e^{-\int_0^t r_s\,ds} \left.\frac{d\mathbb{P}^*}{d\mathbb{P}}\right|_{\mathcal{F}_t} \quad (t \in [0,+\infty))\,.$$

The choice of a specific consumption plan $C \in \mathcal{C}(w)$ will depend on the agent's preferences. A standard approach in the Finance literature consists in restricting attention to the set $\mathcal{C}_{\text{ac}}$ of absolutely continuous consumption plans

$$C_t = \int_0^t c_s\,ds \quad (t \in [0,+\infty))$$

where the progressively measurable process $c = (c_t)_{t\in[0,+\infty)} \geq 0$ specifies a *rate of consumption*. For a time–dependent utility function $u(t,.)$, the problem of finding the best consumption plan C^* in $\mathcal{C}(w) \cap \mathcal{C}_{\text{ac}}$ is then formulated in terms of the utility functional

$$U_{\text{ac}}(C) \stackrel{\Delta}{=} \mathbb{E} \int_0^{+\infty} u(t, c_t)\,dt\,. \tag{15}$$

From a mathematical point of view, this is a space–time version of the standard problem of maximizing expected utility under a linear budget constraint, and its solution is straightforward; see, e.g., [30].

However, as shown in [25], a utility functional of the time–additive form (15) raises serious objections, both from an economic and a mathematical point of view. Firstly, a reasonable extension of the functional U_{ac} from $\mathcal{C}_{\text{ac}}$ to $\mathcal{C}$ only works for spatially affine functions u. Secondly, such functionals are not robust with respect to small time–shifts in consumption plans, and thus do not capture intertemporal substitution effects. Finally, the price functionals arising in the corresponding equilibrium analysis, viewed as continuous linear functionals on the space $\mathcal{C}_{\text{ac}}$ with respect to an L^p–norm on consumption rates, fail to have desirable properties such as the existence of an interest rate. For such reasons, Hindy, Huang and Kreps [25] introduce utility functionals of the following type.

$$U(C) \stackrel{\Delta}{=} \mathbb{E} \int_{(0,+\infty]} u(t, Y_t^C)\,\nu(dt) \quad (C \in \mathcal{C})\,,$$

where ν is a nonnegative optional random measure, and where

$$Y_t^C \stackrel{\Delta}{=} \eta e^{-\beta t} + \int_{[0,t)} \beta e^{-\beta(t-s)}\,dC_s \quad (t \geq 0)$$

serves as an *index of satisfaction*, defined as an exponential average of past consumption. The measure ν accounts for the agent's time preferences. For fixed $t \geq 0$, the utility function $u(t, y)$ is assumed to be strictly concave and increasing in $y \in$

$[0, +\infty)$ with continuous partial derivative $\partial_y u(t, y)$. We assume $\partial_y u(t, 0) \equiv +\infty$, $\partial_y u(t, +\infty) \equiv 0$, and $\partial_y u(., y) \in L^1(\mathbb{P} \otimes \nu)$ for any $y > 0$.

With this choice of preferences, the agent's optimization problem consists in maximizing the concave functional U under a linear constraint:

$$\text{Maximize } U(C) \text{ subject to } C \in \mathcal{C}(w).$$

In [24], this problem is analyzed in a Markovian setting, using the Hamilton–Jacobi–Bellman approach; see also [8].

Let us now describe an alternative approach developed in [6] under the natural assumption that

$$\sup_{C \in \mathcal{C}(w)} U(C) < +\infty \quad \text{for any} \quad w > 0\,.$$

This approach can be applied in a general semimartingale setting, and it leads to a stochastic representation problem of the same type as in the previous section. It is based on the following Kuhn–Tucker criterion for optimality of a consumption plan:

Lemma 1. *A consumption plan $C^* \in \mathcal{C}$ is optimal for its cost*

$$w \stackrel{\Delta}{=} \mathbb{E} \int_{[0,+\infty)} \psi_t \, dC^*_t < +\infty\,,$$

if it satisfies the first order condition

$$\nabla U(C^*) \leq \lambda \psi \quad \text{, with equality } \mathbb{P} \otimes dC^* \text{–a.e.}$$

for some Lagrange multiplier $\lambda > 0$, where the gradient $\nabla U(C^*)$ is defined as the unique optional process such that

$$\nabla U(C^*)_T = \mathbb{E} \left[\int_{(T,+\infty]} \beta e^{-\beta(t-T)} \partial_y u(t, Y_t^{C^*}) \, \nu(dt) \,\middle|\, \mathcal{F}_T \right] \quad \text{for all} \quad T \in \mathcal{T}\,.$$

Proof. Let C^* be as above and take an arbitrary plan $C \in \mathcal{C}(w)$. By concavity we can estimate

$$\begin{aligned} U(C) - U(C^*) &= \mathbb{E} \int_{(0,+\infty]} \{u(t, Y_t^C) - u(t, Y_t^{C^*})\} \, \nu(dt) \\ &\leq \mathbb{E} \int_{(0,+\infty]} \partial_y u(t, Y_t^{C^*}) \{Y_t^C - Y_t^{C^*}\} \, \nu(dt) \\ &= \mathbb{E} \int_{(0,+\infty]} \partial_y u(t, Y_t^{C^*}) \left\{ \int_{[0,t)} \beta e^{-\beta(t-s)} \, (dC_s - dC^*_s) \right\} \nu(dt)\,. \end{aligned}$$

Using Fubini's theorem we thus obtain

$$\begin{aligned} U(C) - U(C^*) &\leq \mathbb{E} \int_{[0,+\infty)} \left\{ \int_{(s,+\infty]} \beta e^{-\beta(t-s)} \partial_y u(t, Y_t^{C^*}) \, \nu(dt) \right\} \\ &\quad \times \, (dC_s - dC^*_s) \\ &= \mathbb{E} \int_{[0,+\infty)} \nabla U(C^*)_s \, (dC_s - dC^*_s) \end{aligned}$$

where the last equality follows from Théorème 1.33 in [27] since $\nabla U(C^*)$ is the optional projection of the $\{\int \dots \nu(dt)\}$–term above. Thus, ∇U serves as a supergradient of U, viewed as a concave functional on the budget set $\mathcal{C}(w)$.

Now, we can use the first order condition to arrive at the estimate

$$U(C) - U(C^*) \leq \lambda \mathbb{E} \int_{[0,+\infty)} \psi_s \,(dC_s - dC^*_s)\,.$$

Since $C \in \mathcal{C}(w)$ and as C^* exhausts the budget w by assumption, the last expectation is ≤ 0, and we can conclude $U(C) \leq U(C^*)$ as desired. □

Combining the first order condition for optimality with a stochastic representation of the price deflator process, we now can describe the optimal consumption plans:

Theorem 5. *Let us assume that for any $\lambda > 0$ the discounted price deflator process $(\lambda e^{-\beta t}\psi_t 1_{[0,+\infty)}(t))_{t\in[0,+\infty]}$ admits a representation*

$$\lambda e^{-\beta T}\psi_T 1_{\{T<+\infty\}} = \mathbb{E}\left[\int_{(T,+\infty]} \beta e^{-\beta t} \partial_y u(t, \sup_{v\in[T,t)} \{L_v e^{\beta(v-t)}\})\, \nu(dt) \,\middle|\, \mathcal{F}_T \right] \tag{16}$$

for $T \in \mathcal{T}$, and for some progressively measurable process $L = (L_t)_{t\geq 0} > 0$ with upper–right continuous paths satisfying

$$\beta e^{-\beta t}\partial_y u(t, \sup_{v\in[T,t)} \{L_v e^{\beta(v-t)}\}) 1_{(T,+\infty]}(t) \in L^1(\mathbb{P}\otimes \nu(dt))$$

for all $T \in \mathcal{T}$.

Then this process L provides a universal consumption signal in the sense that, for any initial level of satisfaction η, the unique plan $C^\eta \in \mathcal{C}$ such that

$$Y^{C^\eta}_t = \eta e^{-\beta t} \vee \sup_{v\in[0,t)} \{L_v e^{\beta(v-t)}\} \quad \textit{for all} \quad t \in (0,+\infty]\,,$$

is optimal for its cost $w = \mathbb{E}\int_{[0,+\infty)} \psi_t \, dC^\eta_t$.

Thus, the optimal consumption plan consists in consuming just enough to ensure that the induced level of satisfaction Y^{C^η} stays above the signal process L which appears in the representation (16) of the price deflator process ψ. This is illustrated in Figure 2.2.

Remark 8. 1. In case μ is atomless and has full support almost surely, existence and uniqueness of the process L appearing in (16) follows from a general representation theorem which will be proved in the next section; cf. Corollary 3.

2. As pointed out in [6], a solution L to the representation problem (16) can be viewed as a minimal level of satisfaction which the agent is willing to accept. Indeed, as shown in Lemma 2.9 of [6], we can represent the process C^η defined in the preceding theorem in the form

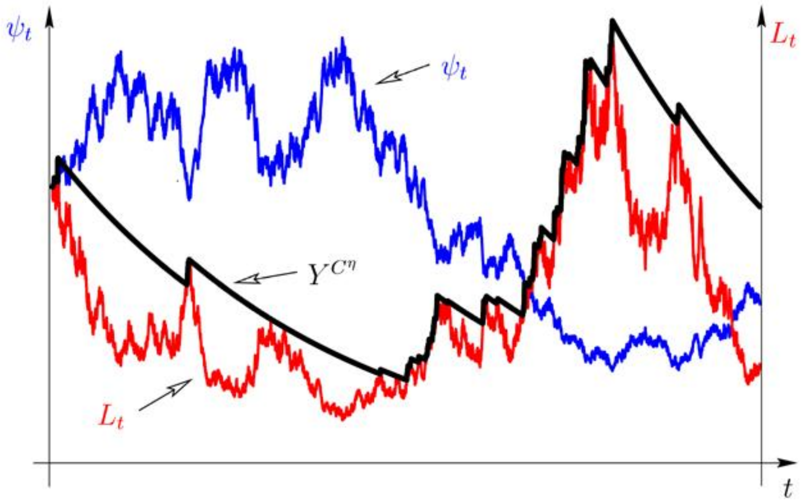

Fig. 2. Typical paths for the deflator ψ (blue or dark gray line), a universal consumption signal L (red or light gray line), and the induced level of satisfaction Y^{C^η} (solid black line).

$$dC_t^\eta = \frac{e^{-\beta t}}{\beta} \, dA_t^\eta \quad (t \in [0, +\infty)) \tag{17}$$

with $A_t^\eta \triangleq \eta \vee \sup_{v \in [0,t]} \{L_v e^{\beta v}\}$ $(t \in [0, +\infty))$. Hence, if $T \in \mathcal{T}$ is a point of increase for C^η, then it is a point of increase for A^η and we have

$$Y_{T+}^{C^\eta} = e^{-\beta T} A_T^\eta = L_T$$

at any such time, while otherwise $Y_{t+}^{C^\eta} = e^{-\beta t} A_t^\eta \geq L_t$.

Proof. We show that the plan $C^* \triangleq C^\eta$ with the above properties satisfies the first order condition

$$\nabla U(C^*) \leq \lambda \psi \quad \text{, with equality } \mathbb{P} \otimes dC^*\text{–a.e.,}$$

of Lemma 1. Indeed, for any $T \in \mathcal{T}$ we have by definition of C^* and monotonicity of $\partial_y u(t, .)$:

$$\begin{aligned} \nabla U(C^*)_T &= \mathbb{E}\left[\int_{(T,+\infty]} \beta e^{-\beta(t-T)} \partial_y u(t, Y_t^{C^*}) \, \nu(dt) \,\middle|\, \mathcal{F}_T \right] \\ &\leq \mathbb{E}\left[\int_{(T,+\infty]} \beta e^{-\beta(t-T)} \partial_y u(t, \sup_{v \in [T,t)} \{L_v e^{\beta(v-t)}\}) \, \nu(dt) \,\middle|\, \mathcal{F}_T \right] \end{aligned} \tag{18}$$

It now follows from the representation property of L that the last conditional expectation is exactly $\lambda\psi_T 1_{\{T<+\infty\}}$. Since $T \in \mathcal{T}$ was arbitrary, this implies $\nabla U(C^*) \leq \lambda\psi$. In order to prove that equality holds true $\mathbb{P}\otimes dC^*$–a.e. let us consider an arbitrary point of increase for C^*, i.e., a stopping time T so that $C^*_{T-} < C^*_t$ for all $t \in (T, +\infty)$ almost surely on $\{0 < T < +\infty\}$. By definition of C^* we obtain

$$Y_t^{C^*} = \sup_{v\in[T,t)} \{L_v e^{\beta(v-t)}\} \quad \text{for any} \quad t \in (T, +\infty] \quad \mathbb{P}\text{–a.s.}.$$

Thus, (18) becomes an equality for any such T. It follows that $\nabla U(C^*) = \lambda\psi$ holds true $\mathbb{P} \otimes dC^*$–a.e., since the points of increase of C^* carry the measure dC^*. □

2.3 Multi–armed Bandits and Gittins Indices

In the multi–armed bandit problem, a gambler faces a slot machine with several arms. All arms yield a payoff of 0 or 1 Euro when pulled, but they may have different payoff probabilities. These probabilities are unknown to the gambler, but playing with the slot machine will allow her to get an increasingly more accurate estimate of each arm's payoff probability. The gambler's aim is to choose a sequence of arms to pull so as to maximize the expected sum of discounted rewards. This choice involves a tradeoff: On the one hand, it seems attractive to pull arms with a currently high estimate of their success probability, on the other hand, one may want to pull other arms to improve the corresponding estimate. In its general form, the multi–armed bandit problem amounts to a dynamic allocation problem where a limited amount of effort is allocated to a number of independent projects, each generating a specific stochastic reward proportional to the effort spent on it.

Gittins' crucial idea was to introduce a family of simpler benchmark problems and to define a dynamic performance measure—now called the Gittins index—separately for each of the projects in such a way that an optimal schedule can be specified as an index–rule: "Always spent your effort on the projects with currently maximal Gittins index". See [23] and [36] for the solution in a discrete–time Markovian setting, [28] and [32] for an analysis of the diffusion case, and [17] and [19] for a general martingale approach.

To describe the connection between the Gittins index and the representation problems discussed in the preceding sections, let us review the construction of Gittins indices in continuous time. Consider a project whose reward is specified by some rate process $(h_t)_{t\in[0,+\infty)}$. With such a project, El Karoui and Karatzas [17] associate the family of optimal stopping problems

$$V_S^m \triangleq \operatorname*{ess\,sup}_{T\in\mathcal{T}([S,+\infty])} \mathbb{E}\left[\int_S^T e^{-\alpha(t-S)} h_t\, dt + m e^{-\alpha(T-S)} \,\middle|\, \mathcal{F}_S\right], \tag{19}$$

for $S \in \mathcal{T},\ m \geq 0$. The optimization starts at time S, the parameter $m \geq 0$ is interpreted as a reward–upon–stopping, and $\alpha > 0$ is a constant discount rate.

Under appropriate conditions, El Karoui and Karatzas [17] show that the Gittins index M of a project can be described as the minimal reward–upon–stopping such that immediate termination of the project is optimal in the auxiliary stopping problem (19), i.e.:

$$M_s = \inf\{m \geq 0 \mid V^m_s = m\} \quad (s \geq 0)\,. \tag{20}$$

They also note in their identity (3.7) that M can be related to the reward process (h_t) via

$$\mathbb{E}\left[\int_s^{+\infty} e^{-\alpha t} h_t\,dt \,\middle|\, \mathcal{F}_s\right] = \mathbb{E}\left[\int_s^{+\infty} \alpha e^{-\alpha t} \sup_{s \leq v \leq t} M_v\,dt \,\middle|\, \mathcal{F}_s\right] \tag{21}$$

for $s \geq 0$. Thus, the Gittins index process M can be viewed as the solution to a representation problem of the form (1). In [17], formula (21) is stated in passing, without making further use of it. Here, we focus on the stochastic representation problem and use it as our starting point. Our main purpose is to emphasize its intrinsic mathematical interest and its unifying role for a number of different applications. In this perspective, formula (20) provides a key to proving existence of a solution to the representation problem in its general form (1), as explained in the next section.

3 A Stochastic Representation Theorem

The previous section has shown how a variety of optimization problems can be reduced to a stochastic representation of a given optional process in terms of running suprema of another process. Let us now discuss the solution of this representation problem from a general point of view.

3.1 The Result and its Application

Let μ be a nonnegative optional random measure and let $f = f(\omega, t, x) : \Omega \times [0, +\infty] \times \mathbb{R} \to \mathbb{R}$ be a random field with the following properties:

1. For any $x \in \mathbb{R}$, the mapping $(\omega, t) \mapsto f(\omega, t, x)$ defines a progressively measurable process in $L^1(\mathbb{P}(d\omega) \otimes \mu(\omega, dt))$.
2. For any $(\omega, t) \in \Omega \times [0, +\infty]$, the mapping $x \mapsto f(\omega, t, x)$ is continuous and strictly decreasing from $+\infty$ to $-\infty$.

Then we can formulate the following general

Representation Problem 1 *For a given optional process $X = (X_t)_{t \in [0,+\infty]}$ with $X_{+\infty} = 0$, construct a progressively measurable process $\xi = (\xi_v)_{v \in [0,+\infty)}$ with upper–right continuous paths such that*

$$f(t, \sup_{v \in [T,t)} \xi_v) 1_{(T,+\infty]}(t) \in L^1(\mathbb{P} \otimes \mu(dt))$$

and

$$X_T = \mathbb{E}\left[\int_{(T,+\infty]} f(t, \sup_{v \in [T,t)} \xi_v)\,\mu(dt) \,\middle|\, \mathcal{F}_T\right]$$

for any stopping time $T \in \mathcal{T}$.

This problem is solved by the following result from [5]. Its proof will be discussed in the next section.

Theorem 6. *If the measure* μ *has full support* $\operatorname{supp}\mu = [0,+\infty]$ *almost surely and* X *is lower-semicontinuous in expectation, then the solution* ξ *to representation problem* (1) *is uniquely determined up to optional sections in the sense that*

$$\xi_S = \operatorname*{ess\,inf}_{T\in\mathcal{T}((S,+\infty])} \varXi_{S,T} \quad \textit{for any} \quad S \in \mathcal{T}([0,+\infty)) \tag{22}$$

where $\varXi_{S,T}$ *denotes the unique* $\mathcal{F}_S$*–measurable random variable satisfying*

$$\mathbb{E}\left[X_S - X_T \,|\, \mathcal{F}_S\right] = \mathbb{E}\left[\int_{(S,T]} f(t,\varXi_{S,T})\,\mu(dt)\,\middle|\,\mathcal{F}_S\right]. \tag{23}$$

If, in addition, μ *almost surely has no atoms, then there actually exists a solution to problem* (1)*.*

Remark 9. If μ has full support almost surely, we have existence and uniqueness of $\varXi_{S,T} \in L^0(\mathcal{F}_S)$ with (23) for any $S \in \mathcal{T}([0,+\infty))$ and any $T \in \mathcal{T}((S,+\infty])$. Indeed, the right side of (23) is then continuous and strictly decreasing in $\varXi = \varXi_{S,T}$ with upper and lower limit $\pm\infty$, respectively. This follows from the corresponding properties of $f = f(\omega,t,x)$ and from the fact that μ has full support.

As an application of Theorem 6, we now can solve all the existence problems arising in our discussion of American put and call options and of optimal consumption plans. This completes the proofs of Theorem 3, Theorem 4. In the context of Theorem 5, this shows that lower–semicontinuity in expectation of the discounted deflator is sufficient for existence of a representation as in (16) if the time–preference measure ν is atomless and has full support almost surely.

Corollary 3. *There exist solutions to the representation problems* (11)*,* (13)*, and* (16)*.*

Proof.

1. For solving the representation problem (11) which characterizes the universal exercise signal for American put options on $(P_t)_{t\in[0,+\infty)}$, we choose $\mu(dt) = re^{-rt}\,dt$ and $f(t,x) \stackrel{\Delta}{=} -x$. Furthermore, we extend $(e^{-rt}P_t)_{t\in[0,\hat{T}]}$ to an optional process X on $[0,+\infty]$ with $X_{+\infty} = 0$:

   $$X_s \stackrel{\Delta}{=} e^{-rs}P_{s\wedge\hat{T}} = \int_{(s,+\infty]} re^{-rt}P_{s\wedge\hat{T}}\,dt \quad (s\in[0,+\infty))\,.$$

   This process is lower–semicontinuous in expectation, due to our assumptions on the process P.
   Applying Theorem 6, we obtain a progressively measurable process ξ with upper–right continuous paths such that

$$
\begin{aligned}
X_T &= \mathbb{E}\left[\int_{(T,+\infty]} f(t, \sup_{v \in [T,t)} \xi_v)\, \mu(dt) \,\middle|\, \mathcal{F}_T \right] \\
&= -\mathbb{E}\left[\int_{(T,+\infty]} re^{-rt} \sup_{v \in [T,t)} \xi_v \, dt \,\middle|\, \mathcal{F}_T \right]
\end{aligned}
$$

for all $T \in \mathcal{T}$. Hence, $K \stackrel{\Delta}{=} -\xi$ is lower–right continuous and satisfies

$$
X_T = \mathbb{E}\left[\int_{(T,+\infty]} re^{-rt} \inf_{v \in [T,t)} K_v \, dt \,\middle|\, \mathcal{F}_T \right] \tag{24}
$$

for any $T \in \mathcal{T}$. Comparing this representation with our definition of X on $[\hat{T}, +\infty]$, we obtain by uniqueness that $\inf_{T \leq v < t} K_v = P_{\hat{T}}$ for any $t > T \geq \hat{T}$. In particular, it follows that $K_{\hat{T}} = P_{\hat{T}}$ by lower–right continuity of K. For stopping times $T \in \mathcal{T}([0, \hat{T}])$, expression (24) therefore transforms into

$$
\begin{aligned}
X_T &= \mathbb{E}\left[\int_{(T,\hat{T}]} re^{-rt} \inf_{v \in [T,t)} K_v \, dt + \int_{(\hat{T},+\infty]} re^{-rt} \inf_{v \in [T,\hat{T})} K_v \wedge P_{\hat{T}} \, dt \,\middle|\, \mathcal{F}_T \right] \\
&= \mathbb{E}\left[\int_{(T,\hat{T}]} re^{-rt} \inf_{v \in [T,t)} K_v \, dt + e^{-r\hat{T}} \inf_{v \in [T,\hat{T}]} K_v \,\middle|\, \mathcal{F}_T \right] .
\end{aligned}
$$

Hence, K solves the representation problem (11).

2. The representation problem (13) for American call options can be solved by applying analogous arguments to the process

$$
X_s \stackrel{\Delta}{=} -e^{-rs} P_{s \wedge \hat{T}} \quad (s \in [0, +\infty]) .
$$

3. For the representation problem (16) which arises in the context of intertemporal consumption choice, we choose $\mu(dt) \stackrel{\Delta}{=} \beta e^{-\beta t} \nu(dt)$,

$$
f(t,x) \stackrel{\Delta}{=} \begin{cases} \partial_y u(t, -e^{-\beta t}/x), & x < 0 \\ -x, & x \geq 0 \end{cases}
$$

and $X_t \stackrel{\Delta}{=} \lambda e^{\beta t} \psi_t 1_{[0,+\infty)}(t)$ $(t \geq 0)$,
Then X, μ, and f satisfy all the assumptions of Theorem 6, and so we obtain a progressively measurable process ξ with upper–right continuous paths such that

$$
\lambda e^{-\beta T} \psi_T 1_{\{T < +\infty\}} = \mathbb{E}\left[\int_{(T,+\infty]} f(t, \sup_{v \in [T,t)} \xi_v)\, \mu(dt) \,\middle|\, \mathcal{F}_T \right]
$$

for any stopping time $T \in \mathcal{T}$. We shall show below that $\xi < 0$ on $[0, +\infty)$ almost surely. Thus, the preceding equation reduces to

$$\lambda e^{-\beta T}\psi_T 1_{\{T<+\infty\}} = \mathbb{E}\left[\int_{(T,+\infty]} \beta e^{-\beta t}\partial_y u(t, -e^{-\beta t}/\sup_{v\in[T,t)} \xi_v)\,\nu(dt)\,\middle|\,\mathcal{F}_T\right].$$

Hence, the representation problem (16) is solved by the process $\{L_v\}_{v\in[0,+\infty)}$ defined by $L_v \stackrel{\Delta}{=} -1/(\xi_v e^{\beta v})$.

In order to prove our claim that $\xi < 0$ on $[0,+\infty)$ almost surely, consider the stopping time

$$\tilde{T} \stackrel{\Delta}{=} \inf\{t \geq 0 \mid \xi_t \geq 0\}.$$

On $\{\tilde{T} < +\infty\}$ upper right continuity of ξ implies $\xi_{\tilde{T}} \geq 0$ almost surely. Thus, choosing $T = \tilde{T}$ in the above representation, we obtain by definition of f:

$$\lambda e^{-\beta\tilde{T}}\psi_{\tilde{T}} 1_{\{\tilde{T}<+\infty\}} = -\mathbb{E}\left[\int_{(\tilde{T},+\infty]} 0 \vee \sup_{v\in[\tilde{T},t)} \xi_v\,\mu(dt)\,\middle|\,\mathcal{F}_{\tilde{T}}\right].$$

Obviously, the right side in this equality is ≤ 0 almost surely while its left side is > 0 except on $\{\tilde{T} = +\infty\}$ where it is 0. It follows that $\mathbb{P}[\tilde{T} = +\infty] = 1$, i.e., $\xi < 0$ on $[0,+\infty)$ $\mathbb{P}$–a.s.. □

In order to illustrate the role of the representation theorem, let us have a closer look at the case of an American put option as discussed in Theorem 3. The decision to exercise an American put option involves a tradeoff between the sure proceeds one can realize immediately and the uncertain future prospects offered by the option. This tradeoff is determined by two factors. Firstly, one has to account for the downward risk in the future evolution of the underlying: If the price process is likely to fall in the near future, one would prefer to wait and exercise the option later. Secondly, one faces a deadline: The option holder can only benefit from the option up to its maturity $\hat{T}$, and so waiting for lower prices bears the risk of not being able to exercise the option at all. The tradeoff between these competing aspects of American puts is reflected in the following characterization of the universal exercise signal $K = (K_t)_{t\in[0,\hat{T}]}$ which is derived from Theorem 6. In fact, for American puts in a model with constant interest rates $r > 0$, the characterization (22) and the arguments for Corollary 3 yield that

$$\begin{aligned} K_S &= \operatorname*{ess\,sup}_{T\in\mathcal{T}((S,+\infty])} \frac{\mathbb{E}\left[e^{-rS}P_S - e^{-rT}P_{T\wedge\hat{T}}\,\middle|\,\mathcal{F}_S\right]}{\mathbb{E}\left[\int_{(S,T]} re^{-rt}\,dt\,\middle|\,\mathcal{F}_S\right]} \\ &= \operatorname*{ess\,sup}_{T\in\mathcal{T}((S,+\infty])} \frac{\mathbb{E}\left[P_S - e^{-r(T-S)}P_{T\wedge\hat{T}}\,\middle|\,\mathcal{F}_S\right]}{\mathbb{E}\left[1 - e^{-r(T-S)}\,\middle|\,\mathcal{F}_S\right]} \end{aligned} \tag{25}$$

for all stopping times $S \in \mathcal{T}([0,\hat{T}])$. It follows that $K_S > k$ iff there is a stopping time $T > S$ such that

$$\mathbb{E}\left[P_S - e^{-r(T-S)}P_{T\wedge\hat{T}}\,\middle|\,\mathcal{F}_S\right] > k\mathbb{E}\left[1 - e^{-r(T-S)}\,\middle|\,\mathcal{F}_S\right]$$

or, equivalently,

$$k - P_S < \mathbb{E}\left[e^{-r(T-S)}(k - P_{T\wedge\hat{T}}) \,\middle|\, \mathcal{F}_S \right] \le \mathbb{E}\left[e^{-r(T-S)}(k - P_{T\wedge\hat{T}})^+ \,\middle|\, \mathcal{F}_S \right] .$$

Hence, $K_S > k$ means that exercising the put option with strike k should be postponed since there is an opportunity for stopping later than S which makes us expect a higher discounted payoff. This provides another intuitive explanation why K_S should be viewed as a universal exercise signal. However, using formula (25) in order to compute K_S amounts to solving a non–standard optimal stopping problem for a quotient of two expectations. Such stopping problems are hard to solve directly. Moritomo [34] uses a Lagrange multiplier technique in order to reduce this non–standard problem to the solution of a family of standard optimal stopping problems. In the context of American options, this is as complex as the initially posed problem of optimally exercising the American put with arbitrary strike. In contrast, our characterization of K_S via the representation problem (1) provides a possibility to compute K_S without solving any optimal stopping problems, as illustrated by the case studies in Section 4.

3.2 Proof of Existence and Uniqueness

Let us now discuss the proof of Theorem 6, following the arguments of [5]. We start with the uniqueness part and prove the characterization

$$\xi_S = \operatorname*{ess\,inf}_{T\in\mathcal{T}((S,+\infty])} \Xi_{S,T} \quad \text{for any} \quad S \in \mathcal{T}([0,+\infty)) \tag{26}$$

with $\Xi_{S,T}$ as in (23). In order to show that '$\le$' holds true, consider a stopping time $T \in \mathcal{T}((S,+\infty])$ and use the representation property of ξ to write

$$X_S = \mathbb{E}\left[\int_{(S,T]} f(t, \sup_{v\in[S,t)} \xi_v)\,\mu(dt) \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[\int_{(T,+\infty]} f(t, \sup_{v\in[S,t)} \xi_v)\,\mu(dt) \,\middle|\, \mathcal{F}_S \right] .$$

As $f(t,.)$ is decreasing, the first $f(\ldots)$–term is $\le f(t,\xi_S)$ and the second one is $\le f(t, \sup_{v\in[T,t)} \xi_v)$. Hence:

$$X_S \le \mathbb{E}\left[\int_{(S,T]} f(t,\xi_S)\,\mu(dt) \,\middle|\, \mathcal{F}_S \right] + \mathbb{E}\left[\int_{(T,+\infty]} f(t, \sup_{v\in[T,t)} \xi_v)\,\mu(dt) \,\middle|\, \mathcal{F}_S \right] .$$

Using the representation property of ξ again, we can rewrite the second conditional expectation as

$$\mathbb{E}\left[\int_{(T,+\infty]} f(t, \sup_{v\in[T,t)} \xi_v)\,\mu(dt) \,\middle|\, \mathcal{F}_S \right] = \mathbb{E}\left[X_T \,|\, \mathcal{F}_S \right] .$$

It follows by definition of $\Xi_{S,T}$ that

$$\mathbb{E}\left[\int_{(S,T]} f(t,\Xi_{S,T})\,\mu(dt)\,\middle|\,\mathcal{F}_S\right] = \mathbb{E}\left[X_S - X_T \,|\, \mathcal{F}_S\right]$$
$$\leq \mathbb{E}\left[\int_{(S,T]} f(t,\xi_S)\,\mu(dt)\,\middle|\,\mathcal{F}_S\right].$$

As both $\Xi_{S,T}$ and ξ_S are $\mathcal{F}_S$–measurable, this implies that $\xi_S \leq \Xi_{S,T}$ almost surely.

In order to show that ξ_S is the largest larger lower bound on the family $\Xi_{S,T},\ T \in \mathcal{T}((S,+\infty])$, consider the sequence of stopping times

$$T^n \triangleq \inf\left\{t \in (S,+\infty]\,\middle|\,\sup_{v\in[S,t)} \xi_v > \eta_n\right\} \quad (n = 1,2,\dots)$$

where

$$\eta_n = (\xi_S + 1/n)1_{\{\xi_S > -\infty\}} - n1_{\{\xi_S = -\infty\}}.$$

Observe that pathwise upper–right continuity of ξ implies $T^n \in \mathcal{T}((S,+\infty])$ and also

$$\sup_{v\in[S,t)} \xi_v = \sup_{[T^n,t)} \xi_v \quad \text{for all} \quad t \in (T^n,+\infty] \quad \mathbb{P}\text{–a.s.}$$

since T^n is a time of increase for $t \mapsto \sup_{v\in[S,t)} \xi_v$. Thus, we obtain

$$X_S = \mathbb{E}\left[\int_{(S,T^n]} f(t, \sup_{v\in[S,t)} \xi_v)\,\mu(dt)\,\middle|\,\mathcal{F}_S\right]$$
$$+ \mathbb{E}\left[\int_{(T^n,+\infty]} f(t, \sup_{v\in[T^n,t)} \xi_v)\,\mu(dt)\,\middle|\,\mathcal{F}_S\right]$$
$$\geq \mathbb{E}\left[\int_{(S,T^n]} f(t,\eta_n)\,\mu(dt)\,\middle|\,\mathcal{F}_S\right] + \mathbb{E}\left[X_{T^n}\,|\,\mathcal{F}_S\right],$$

where the last estimate follows from our definition of T^n and from the representation property of ξ at time T^n. As η_n is $\mathcal{F}_S$–measurable, the above estimate implies

$$\eta_n \geq \Xi_{S,T^n} \geq \operatorname*{ess\,inf}_{T\in\mathcal{T}((S,+\infty])} \Xi_{S,T}.$$

Now note that for $n \uparrow +\infty$, we have $\eta_n \downarrow \xi_S$, and so we obtain the converse inequality '$\geq$' in (26).

Let us now turn to the existence part of Theorem 6, and let us sketch the construction of a solution ξ to the representation problem 1; for the technical details we refer to [5].

The definition of Gittins indices (20) and their representation property (21) suggest to consider the family of optimal stopping problems

$$Y_S^x = \operatorname*{ess\,inf}_{T\in\mathcal{T}([S,+\infty])} \mathbb{E}\left[X_T + \int_{(S,T]} f(t,x)\,\mu(dt)\,\middle|\,\mathcal{F}_S\right] \quad (S \in \mathcal{T},\ x \in \mathbb{R}) \quad (27)$$

and to define the process ξ as

$$\xi_t(\omega) \stackrel{\Delta}{=} \max\{x \in \mathbb{R} \cup \{-\infty\} \mid Y_t^x(\omega) = X_t(\omega)\} \quad (t \in [0, +\infty), \omega \in \Omega)\,. \quad (28)$$

Since μ has no atoms, we can use results from [16] to choose a 'nice' version of the random field $Y = (Y_S^x)$ such that ξ is an optional process and such that for any $x \in \mathbb{R}$, $S \in \mathcal{T}$ the stopping time

$$T_S^x \stackrel{\Delta}{=} \inf\{t \geq S \mid Y_t^x = X_t\} \in \mathcal{T}([S, +\infty])$$

attains the essential infimum in (27):

$$Y_S^x = \mathbb{E}\left[X_{T_S^x} + \int_{(S, T_S^x]} f(t, x)\, \mu(dt) \,\middle|\, \mathcal{F}_S \right]. \quad (29)$$

For any $S \in \mathcal{T}$, Y_S^x is dominated by X_S and continuously decreasing in x with $\lim_{x \downarrow -\infty} Y_S^x = X_S$ almost surely. The key observation is that the corresponding negative random measure $Y_S(dx)$ can be disintegrated in the form

$$Y_S(dx) = \mathbb{E}\left[\int_{(S, +\infty]} \left\{ \int_{-\infty}^{+\infty} 1_{(S, T_S^x]}(t)\, f(t, dx) \right\} \mu(dt) \,\middle|\, \mathcal{F}_S \right],$$

where $f(t, dx)$ is the negative measure induced by the decreasing function $x \mapsto f(t, x)$. This disintegration formula can be viewed as a generalization of Lemma 2.3 in [17] to the nonlinear case; compare also Lemma 2 in [36] for a discrete–time analogue in a Markovian setting.

Using the definition of ξ_S, this allows us to write for any $y \in \mathbb{R}$:

$$\begin{aligned} X_S = Y_S^{\xi_S} &= Y_S^y - \int_{\xi_S \wedge y}^{y} Y_S(dx) \\ &= Y_S^y - \mathbb{E}\left[\int_{(S, +\infty]} \left\{ \int_{-\infty}^{+\infty} 1_{(S, T_S^x]}(t) 1_{[\xi_S \wedge y, y)}(x) f(t, dx) \right\} \mu(dt) \,\middle|\, \mathcal{F}_S \right] \end{aligned}$$

By definition of T_S^x and ξ, we have

$$\begin{aligned} \{(\omega, t, x) \mid T_S^x \geq t\} &= \{(\omega, t, x) \mid Y_v^x < X_v \text{ for all } v \in [S, t)\} \\ &= \{(\omega, t, x) \mid x > \xi_v \text{ for all } v \in [S, t)\} \\ &= \{(\omega, t, x) \mid x \geq \sup_{v \in [S, t)} \xi_v\} \end{aligned}$$

up to a $\mathbb{P} \otimes \mu(dt) \otimes f(t, dx)$–null set. Hence, the above conditional expectation simplifies to

$$\mathbb{E}\left[\int_{(S,+\infty]}\left\{\int_{-\infty}^{+\infty} 1_{(S,T_S^x]}(t)1_{[\xi_S\wedge y,y)}(x)f(t,dx)\right\}\mu(dt)\,\middle|\,\mathcal{F}_S\right]$$
$$=\mathbb{E}\left[\int_{(S,+\infty]}\left\{\int_{-\infty}^{+\infty} 1_{[\sup_{v\in[S,t)}\xi_v,+\infty)}(x)1_{[\xi_S\wedge y,y)}(x)f(t,dx)\right\}\mu(dt)\,\middle|\,\mathcal{F}_S\right]$$
$$=\mathbb{E}\left[\int_{(S,+\infty]}\left\{f(t,y)-f(t,\sup_{v\in[S,t)}\xi_v\wedge y)\right\}\mu(dt)\,\middle|\,\mathcal{F}_S\right]$$
$$=\mathbb{E}\left[\int_{(S,T_S^x]}\left\{f(t,y)-f(t,\sup_{v\in[S,t)}\xi_v)\right\}\mu(dt)\,\middle|\,\mathcal{F}_S\right]$$

where the last equality holds true since

$$f(t,y)=f(t,\sup_{v\in[S,t)}\xi_v\wedge y)\ \text{ on }\ \{\sup_{v\in[S,t)}\xi_v>y\}=\{T_S^y<t\}.$$

Plugging this equation into the above representation of X_S we obtain

$$X_S=Y_S^y-\mathbb{E}\left[\int_{(S,T_S^y]}\left\{f(t,y)-f(t,\sup_{v\in[S,t)}\xi_v)\right\}\mu(dt)\,\middle|\,\mathcal{F}_S\right] \tag{30}$$
$$=\mathbb{E}\left[X_{T_S^y}\,\middle|\,\mathcal{F}_S\right]+\mathbb{E}\left[\int_{(S,T_S^y]}f(t,\sup_{v\in[S,t)}\xi_v)\,\mu(dt)\,\middle|\,\mathcal{F}_S\right]$$

where the second equality follows from (29) for $x=y$. Letting $y\uparrow+\infty$ in (30), we deduce the desired representation

$$X_S=\mathbb{E}\left[\int_{(S,+\infty]}f(t,\sup_{v\in[S,t)}\xi_v)\,\mu(dt)\,\middle|\,\mathcal{F}_S\right].$$

4 Explicit Solutions

Let us now provide explicit solutions to the representation problem discussed in the previous section in some specific models with strong homogeneity properties.

4.1 Lévy Models

In this section, we consider two situations where the source of randomness is modelled as a Lévy process $Y=(Y_t)_{t\in[0,+\infty)}$, defined as a right continuous process whose increments Y_t-Y_s, $s\le t$, are independent of $\mathcal{F}_s$ and have the same distribution as Y_{t-s}; see [9]. As classical examples, this includes Brownian motions and Poisson processes with constant drift. But there is a rich variety of other Lévy models appearing in Finance; see, e.g., [15], [7].

The Perpetual American Put

We shall start our illustration by considering a perpetual American put on an underlying process P which takes the form

$$P_t = p\exp(Y_t) \quad (t \geq 0) \tag{31}$$

for some initial price $p > 0$ and some Lévy process Y. Let us assume that interest rates are given by a constant $r > 0$. In this case, Theorem 3 suggests to consider the representation problem

$$e^{-rT}P_T = \mathbb{E}\left[\int_{(T,+\infty]} re^{-rt} \inf_{v\in[T,t)} K_v \, dt \,\middle|\, \mathcal{F}_T\right] \quad (T \in \mathcal{T}). \tag{32}$$

This problem can be solved explicitly:

Lemma 2. *The process* $K_v = P_v/\kappa$ $(v \geq 0)$ *with*

$$\kappa \stackrel{\Delta}{=} \mathbb{E}\left[\int_{(0,+\infty]} re^{-rt} \inf_{v\in[0,t)} \exp(Y_v)\, dt\right] \in (0,1)$$

solves the representation problem (32) *for the perpetual American put.*

Proof. Take a stopping time $T \in \mathcal{T}([0,+\infty])$, and use the Ansatz $K_v = P_v/\kappa$ with $\kappa > 0$ to rewrite the right side of (32) as

$$\begin{aligned}
&\mathbb{E}\left[\int_{(T,+\infty]} re^{-rt} \inf_{v\in[T,t)} K_v \, dt \,\middle|\, \mathcal{F}_T\right] \\
&= \mathbb{E}\left[\int_{(T,+\infty]} re^{-rt} \inf_{v\in[T,t)} \{p\exp(Y_v)/\kappa\} \, dt \,\middle|\, \mathcal{F}_T\right] \\
&= pe^{-rT}\exp(Y_T)\mathbb{E}\left[\int_{(T,+\infty]} re^{-r(t-T)} \inf_{v\in[T,t)} \exp(Y_v - Y_T) \, dt \,\middle|\, \mathcal{F}_T\right] /\kappa \\
&= e^{-rT}P_T\mathbb{E}\left[\int_{(0,+\infty]} re^{-rt} \inf_{v\in[0,t)} \exp(Y_v) \, dt\right] /\kappa
\end{aligned}$$

where for the last equality we used that Y is a Lévy process. Now, choosing κ as in the formulation of the present lemma yields the solution to (32). □

It follows from Theorem 3 that an investor using (31) as a model for the underlying will exercise a perpetual American put with strike $k > 0$ at time

$$\underline{T}^k = \inf\{t \geq 0 \mid K_t \leq k\} = \inf\{t \geq 0 \mid P_t \leq \kappa k\}.$$

i.e., when the underlying's price has fallen below $100 \times \kappa\%$ of the strike. This result also appears in [33], but the proof is different. It reduces the problem to a classical result on optimal stopping rules for geometric random walks by [11]. See also [1] and [10].

Optimal Consumption

In the context of optimal consumption choice as discussed in Section 2.2 and under appropriate homogeneity assumptions, the arguments for obtaining Lemma 2 yield an explicit representation for the discounted price deflator $(e^{-\beta t}\psi_t)_{t\in[0,+\infty)}$. In fact, suppose that the deflator ψ takes the form of an exponential Lévy process,

$$\psi_t = \exp(Y_t) \quad (t \in [0, +\infty))\,,$$

and that the agent's utility function $u(t, y)$ is constant over time and of the HARA form

$$u(t, y) = \frac{y^\alpha}{\alpha} \quad (t, y \in [0, +\infty))$$

for some parameter of risk aversion $\alpha \in (0, 1)$. Furthermore, assume a homogeneous time preference structure specified by $\nu(dt) \stackrel{\Delta}{=} \delta e^{-\delta t}\, dt$ for some constant $\delta > 0$. Then the representation problem (16) of Theorem 5 reads

$$\begin{aligned}\lambda e^{-\beta T}\psi_T &= \mathbb{E}\left[\int_{(T,+\infty]} \partial_y u(\sup_{v\in[T,t)}\{L_v e^{\beta(v-t)}\})\beta\delta e^{-(\beta+\delta)t}\, dt \,\middle|\, \mathcal{F}_T\right]\\ &= \mathbb{E}\left[\int_{(T,+\infty]} \delta e^{-(\alpha\beta+\delta)t} \inf_{v\in[T,t)}\{L_v^{\alpha-1}e^{\beta(\alpha-1)v}\}\, dt \,\middle|\, \mathcal{F}_T\right]\end{aligned}$$

with $T \in \mathcal{T}$. Since ψ is an exponential Lévy process, this is essentially the same representation problem as discussed in Lemma 2. We can therefore identify the solution to (16) as the process L given by

$$L_v = \left(e^{\delta t}\psi_t\right)^{-\frac{1}{1-\alpha}} / \kappa \quad (v \in [0, +\infty))$$

for some constant $\kappa > 0$. Hence, the minimal level process is again an exponential Lévy process. It now follows from the description of optimal consumption plans given in Theorem 5 and Equation (17) that the qualitative behavior of the consumption process is the same as the behavior of the running supremum of such an exponential Lévy process.

In the economic interpretation, this implies that a variety of different consumption patterns can be optimal, depending on the underlying stochastic model. If, for instance, ψ is an exponential Poisson process with drift, consumption will occur in gulps whenever there is a favorable downward price shock. Consumption at rates occurs in models where the deflator is driven by a Lévy process without downward jumps and with vanishing diffusion part. If, on the other hand, the price deflator ψ is specified as a geometric Brownian motion, consumption occurs in a singular way, similar to the behavior of Brownian local time. For a more detailed study of optimal consumption behavior, including a discussion of the corresponding investment strategies, we refer to [2] and [6].

4.2 Diffusion Models

Let $X = (X_t)_{t \in [0,+\infty)}$ be specified as a time–homogeneous one–dimensional diffusion with state space $(0,+\infty)$, and let $\mathbb{P}_x$ denote its distribution when started in $x \in (0,+\infty)$. An application of the strong Markov property shows that the Laplace transforms of the level passage times

$$T_y = \inf\{t \geq 0 \mid X_t = y\}\,.$$

satisfy

$$\mathbb{E}_x e^{-rT_z} = \mathbb{E}_x e^{-rT_y} \mathbb{E}_y e^{-rT_z} \quad \text{for any} \quad x > y > z \geq 0,\ r > 0\,.$$

Hence, these Laplace transforms are of the form

$$\mathbb{E}_x e^{-rT_y} = \frac{\varphi_r(x)}{\varphi_r(y)} \quad (x > y > 0) \tag{33}$$

for some continuous and strictly decreasing function $\varphi_r : (0,+\infty) \to (0,+\infty)$ with $\varphi_r(y) \uparrow +\infty$ as $y \downarrow 0$; we refer to [26] for a detailed discussion.

Lemma 3. *If the function φ_r of* (33) *is strictly convex and continuously differentiable, the solution $\xi = (\xi_v)_{v \in [0,+\infty)}$ of the representation problem*

$$e^{-rT} X_T 1_{\{T<+\infty\}} = \mathbb{E}\left[\int_{(T,+\infty]} r e^{-rt} \inf_{v \in [T,t)} \xi_v \, dt \,\middle|\, \mathcal{F}_T \right] \quad (T \in \mathcal{T}) \tag{34}$$

takes the form $\xi_v = \kappa(X_v)$ where the function κ is given by

$$\kappa(x) \overset{\Delta}{=} x - \frac{\varphi_r(x)}{\varphi_r'(x)} \quad (x \in (0,+\infty))\,. \tag{35}$$

Proof. We choose the Ansatz $\xi_v = \kappa(X_v)$, where κ is a continuous function on $(0,+\infty)$. Using the strong Markov property, we see that the representation problem (34) amounts to specifying κ such that

$$x = \mathbb{E}_x \int_0^{+\infty} r e^{-rt} \inf_{v \in [0,t)} \kappa(X_v)\, dt \quad \text{for all} \quad x \in [0,+\infty)\,.$$

Equivalently, we can write

$$x = \mathbb{E}_x \inf_{v \in [0,\tau_r)} \kappa(X_v) \quad \text{for all} \quad x \in [0,+\infty)\,, \tag{36}$$

where τ_r denotes an independent, exponentially distributed random time with parameter r. If we assume that κ is strictly increasing with $\kappa(0+) = 0$ and $\kappa(+\infty) = +\infty$, then the right side in (36) can be rewritten as

$$\mathbb{E}_x\left[\inf_{v\in[0,\tau_r)}\kappa(X_v)\right] = \mathbb{E}_x\left[\kappa(\inf_{v\in[0,\tau_r)}X_v)\right] = \int_0^{+\infty}\mathbb{P}_x\left[\kappa(\inf_{v\in[0,\tau_r)}X_v)>y\right]dy\,. \qquad (37)$$

We have

$$\mathbb{P}_x[\kappa(\inf_{v\in[0,\tau_r)}X_v)>y] = \mathbb{P}_x[\inf_{v\in[0,\tau_r)}X_v>\kappa^{-1}(y)] = \begin{cases} 0 & \text{if } x\le\kappa^{-1}(y),\ \text{i.e., } y\ge\kappa(x),\\ \mathbb{P}_x[T_{\kappa^{-1}(y)}>\tau_r] & \text{otherwise.}\end{cases}$$

By Fubini's theorem,

$$\begin{aligned}\mathbb{P}_x[T_{\kappa^{-1}(y)}>\tau_r] &= \int_0^{+\infty} re^{-rt}\left\{\int_t^{+\infty}\mathbb{P}_x[T_{\kappa^{-1}(y)}\in ds]\right\}dt\\ &= \int_0^{+\infty}\left\{\int_0^s re^{-rt}\,dt\right\}\mathbb{P}_x[T_{\kappa^{-1}(y)}\in ds]\\ &= 1-\mathbb{E}_x e^{-rT_{\kappa^{-1}(y)}} = 1-\frac{\varphi_r(x)}{\varphi_r(\kappa^{-1}(y))}\,.\end{aligned}$$

Plugging this into (37) yields

$$\mathbb{E}_x\left[\inf_{v\in[0,\tau_r)}\kappa(X_v)\right] = \int_0^{\kappa(x)}\left\{1-\frac{\varphi_r(x)}{\varphi_r(\kappa^{-1}(y))}\right\}dy = \kappa(x)-\varphi_r(x)\int_0^x\frac{d\kappa(z)}{\varphi_r(z)}\,,$$

where we use the substitution $y=\kappa(z)$ in the last step. Combining this with (36) shows that κ satisfies

$$\kappa(x) = x+\varphi_r(x)\int_0^x\frac{d\kappa(z)}{\varphi_r(z)} \quad (x\in(0,+\infty))\,.$$

Writing this identity in differential form yields

$$d\kappa(x) = dx + d\varphi_r(x)\int_0^x\frac{d\kappa(z)}{\varphi_r(z)} + d\kappa(x)$$

or, equivalently,

$$d\kappa(x) = -\varphi_r(x)\,d\frac{1}{\varphi_r'(x)}\,.$$

Thus,

$$\kappa(x) = -\int_0^x\varphi_r(y)\,d\frac{1}{\varphi_r'(y)} = x-\frac{\varphi_r(x)}{\varphi_r'(x)} \quad (x\in(0,+\infty))\,,$$

where the last equality follows by partial integration; note that $\lim_{y\downarrow 0}\varphi_r(y)/\varphi_r'(y)=0$ by convexity of φ_r. Since φ_r is strictly convex with continuous derivative by assumption, this function κ is in fact strictly increasing, continuous and surjective. Hence, the preceding calculations are justified, and so we have shown that the function κ defined in (35) satisfies (36) as desired. □

The explicit solution derived in Lemma 3 can readily be applied to the different optimization problems discussed in Section 2. In fact, this result is closely related to the explicit computation of Gittins indices for one–dimensional diffusions as carried out in [28] and [17]. Note, however, that their calculation is based on the characterization of Gittins indices as essential infima over certain stopping times, while our argument identifies the function κ directly as the solution of the representation problem (36).

5 Algorithmic Aspects

Closed–form solutions as derived in the previous sections are typically available only under strong homogeneity assumptions. In practice, however, one usually has to face inhomogeneities. One important example in Finance is the American put with finite time horizon which does not allow for closed–form solutions even in the simplest case of the Black–Scholes model. In order to deal with such inhomogeneous problems, it becomes necessary to use computational methods. For this reason, let us focus on some algorithmic aspects of our general representation problem (1) in a discrete–time setting, following [3].

Specifically, we assume that μ is given as a sum of Dirac measures

$$\mu(dt) = \sum_{i=1}^{n+1} \delta_{t_i}(dt)$$

so that

$$\mathbb{T} \stackrel{\Delta}{=} \operatorname{supp}\mu \cup \{0\} = \{0 \stackrel{\Delta}{=} t_0 < t_1 < \ldots < t_{n+1} \stackrel{\Delta}{=} +\infty\}$$

is finite. Suppose that, for any $t = t_1, \ldots, t_{n+1}$, the function $f = f(\omega, t, x)$ is continuously and strictly decreasing from $+\infty$ to $-\infty$ in $x \in \mathbb{R}$ with

$$f(t,x) \in L^1(\Omega, \mathcal{F}_t, \mathbb{P})\,.$$

In this situation the construction of a solution to the discrete–time version

$$X_T = \mathbb{E}\left[\sum_{s\in\mathbb{T},\ s>T} f(s, \max_{v\in\mathbb{T}\cap[t,s)} \xi_v) \,\middle|\, \mathcal{F}_T\right] \quad (T \in \mathcal{T}(\mathbb{T}))\,. \tag{38}$$

of our representation problem becomes straightforward.

In fact, there are several rather obvious ways to compute the solution $\xi = (\xi_t)_{t\in\mathbb{T}\cap[0,+\infty)}$. One approach is by backwards induction: First solve for ξ_{t_n} in

$$X_{t_n} = \mathbb{E}\left[\, f(t_{n+1}, \xi_{t_n}) \,|\, \mathcal{F}_{t_n}\right]$$

and then, having constructed $\xi_{t_n}, \xi_{t_{n-1}}, \ldots, \xi_{t_{i+1}}$, compute ξ_{t_i} as the unique solution $\Xi \in L^0(\mathcal{F}_{t_i})$ to the equation

$$X_{t_i} = \mathbb{E}\left[f(t_{i+1}, \Xi) + \sum_{s\in\mathbb{T},\ s>t_{i+1}} f(s, \Xi \vee \max_{v\in\mathbb{T}\cap[t_{i+1},s)} \xi_v) \,\middle|\, \mathcal{F}_{t_i} \right].$$

However, this approach may be tedious from a computational point of view. In fact, solving for Ξ in the above equation amounts to solving a highly nonlinear equation given in terms of a conditional expectation of a path–dependent random variable.

As an alternative, one might use the characterization of ξ in Theorem 6 and compute

$$\operatorname*{ess\,inf}_{T\in\mathcal{T}(\mathbb{T}\cap(t_i,+\infty])} \Xi_{t,T}$$

for each $t \in \mathbb{T} \cap [0, +\infty)$, where $\Xi_{t,T}$ denotes the unique solution $\Xi \in L^0(\mathcal{F}_t)$ to

$$\mathbb{E}\left[X_t - X_T \,|\, \mathcal{F}_t\right] = \mathbb{E}\left[\sum_{s\in\mathbb{T},\ s>t} f(s, \Xi) \,\middle|\, \mathcal{F}_t \right].$$

Solving for Ξ in this equation is comparably easy. For instance, in the separable case $f(s,x) = g(s)h(x)$ one finds

$$\Xi_{t,T} = h^{-1}\left(\frac{\mathbb{E}\left[X_t - X_T \,|\, \mathcal{F}_t\right]}{\mathbb{E}\left[\sum_{s\in\mathbb{T},\ s>t} g(s) \,\middle|\, \mathcal{F}_t\right]} \right).$$

A crucial drawback of this approach, however, is that the class of stopping times $\mathcal{T}(\mathbb{T}\cap(t,+\infty])$ is typically huge. Hence, it would be convenient to reduce the number of stopping times T to be considered. This is achieved by the following

Algorithm 2

```
AdaptedProcess ξ; ξ_{+∞} = +∞;
for (int i = n, i ≥ 0, i = i − 1) {
     StoppingTime T = t_{i+1};
     while (P[Ξ_{t_i,T} > ξ_T] > 0) {
          T = min{t ∈ T ∩ (T, +∞] | ξ_t ≥ ξ_T}
               on {ξ_T = F_{t_i}–ess inf ξ_T < Ξ_{t_i,T}};
     };
     ξ_{t_i} = Ξ_{t_i,T};
};
```

Here $\mathcal{F}_t$– ess inf ξ_T denotes the largest $\mathcal{F}_t$–measurable random variable which is almost surely dominated by ξ_T:

$$\mathcal{F}_t\text{– ess inf}\, \xi_T = \operatorname{ess\,sup}\left\{\Xi \in L^0(\mathcal{F}_t) \mid \Xi \le \xi_T \ \ \mathbb{P}\text{–a.s.}\right\}.$$

Like the first approach, the algorithm proceeds backwards in time. Similar to the second approach, it constructs the solution ξ_t, $t = t_n, t_{n-1}, \ldots, t_0$, in the form $\xi_t = \Xi_{t,T}$. However, instead of considering *all* stopping times $T \in \mathcal{T}(t, +\infty]$ in order to determine a stopping time with $\xi_t = \Xi_{t,T}$, the algorithm constructs an increasing sequence of candidates, starting with the first time in $\mathbb{T}$ after t. Step by step, this candidate is carefully updated until the terminal condition $\mathbb{P}[\Xi_{t,T} > \xi_T] = 0$ is met.

It follows from the monotonicity of the update rule for T that the algorithm will terminate under

Assumption 3 *The set of scenarios Ω is finite.*

The main idea of the algorithm is to construct for each $i = n, \dots, 0$ the stopping time

$$S_i^* \stackrel{\Delta}{=} \min\{s \in \mathbb{T} \cap (t_i, +\infty] \mid \xi_s \geq \xi_{t_i}\}\,. \tag{39}$$

Since $\mathbb{T}$ is discrete, this stopping time is contained in $\mathcal{T}(\mathbb{T} \cap (t_i, +\infty])$ and it attains the $\operatorname{ess\,inf}$ in the characterization of ξ_{t_i} provided by Theorem 6:

Lemma 4. *For any $t_i \in \mathbb{T} \cap [0, +\infty)$, we have*

$$\Xi_{t,S_i^*} = \xi_t = \operatorname*{ess\,inf}_{S \in \mathcal{T}(\mathbb{T} \cap (t_i, +\infty])} \Xi_{t,S}\,. \tag{40}$$

Proof. The first equality is established with the same argument as in the '$\geq$'–part of the proof of Theorem 6, choosing $T^n \equiv S_i^* \in \mathcal{T}(\mathbb{T} \cap (t, +\infty])$. The second one follows as in Theorem 6. □

It may seem that the preceding lemma is not of great help for computing ξ_{t_i} since ξ_{t_i} appears in the definition of S_i^*. However, we are going to show that the stopping time attained upon termination of the `while`–loop at stage i coincides with S_i^* even though its construction does not rely on ξ_{t_i}. This will be the main step in our proof of

Theorem 7. *Algorithm 2 is correct: The resulting process ξ solves the representation problem* (38).

From now on we fix the index i and write $S^* = S_i^*$. Our aim is to prove the identity

$$S^* = T^* \quad \mathbb{P}\text{–a.s.} \tag{41}$$

where T^* denotes the value of the algorithm's stopping time T upon termination of the `while`-loop at stage i. As a first step, let us characterize S^* in a different way:

Lemma 5. *The stopping time S^* of* (39) *is minimal among all stopping times $S \in \mathcal{T}(t_i, +\infty]$ satisfying $\xi_S \geq \Xi_{t_i,S}$ almost surely.*

Proof. The inequality $\xi_{S^*} \geq \Xi_{t_i,S^*}$ follows immediately from (39) and (40). On the other hand, (40) entails that, for any $S \in \mathcal{T}(t_i, +\infty]$ with $\xi_S \geq \Xi_{t_i,S}$ almost surely, we have $\xi_S \geq \xi_{t_i}$ almost surely. But this implies $S \geq S^*$ $\mathbb{P}$–a.s. by definition of S^*. □

Let us denote the successive instances of the stopping time T during the procession of the `while`–loop at stage i by $T^0 \stackrel{\Delta}{=} t_{i+1} \leq T^1 \leq \ldots \leq T^*$ with the convention that $T^k = T^*$ if the `while`–loop is processed less than k times. It then follows from the update rule of our algorithm that

$$\{T^k < T^{k+1}\} = \{\xi_{T^k} = \mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k} < \Xi_{t_i,T^k}\} \quad \mathbb{P}\text{–a.s.} \tag{42}$$

Since Ω is finite, the `while`–loop will be terminated at some point. We thus have $T^k = T^{k+1} = T^*$ $\mathbb{P}$–a.s. for all sufficiently large k. By (42) this means that T^* satisfies $\Xi_{t_i,T^*} \leq \xi_{T^*}$ almost surely. In particular, we can infer from Lemma 5 that

$$S^* \leq T^* \quad \mathbb{P}\text{–a.s.}$$

Thus, in order to establish our central claim (41), it remains to prove the converse inequality. This is achieved by

Lemma 6. $T^k \leq S^*$ *almost surely for each* $k = 0, 1, \ldots$.

Proof. Since $T^0 = t_{i+1}$ and $S^* \geq t_{i+1}$ by definition, our assertion holds true for $k = 0$ and so we can proceed by induction. Thus, assume that we already have established $T^k \leq S^*$ and let us deduce that also $T^{k+1} \leq S^*$ almost surely.

To this end, note that on $\{T^k < S^*\}$ we have $\xi_{T^k} < \xi_{t_i} \leq \xi_{S^*}$ by definition of S^*. Since, by definition, T^{k+1} coincides either with T^k or with the first time in $\mathbb{T}$ after T^k where ξ reaches or exceeds the level ξ_{T^k}, this implies

$$T^{k+1} \leq S^* \quad \text{almost surely on} \quad \{T^k < S^*\}\,.$$

Hence, our claim $T^{k+1} \leq S^*$ $\mathbb{P}$–a.s. will be proved once we know that

$$\{T^k < T^{k+1}\} \subset \{T^k < S^*\} \quad \text{up to a } \mathbb{P}\text{–null set.} \tag{43}$$

This inclusion will be established using the following two intermediate results:

1. *Up to a $\mathbb{P}$–null set we have* $\{\mathbb{P}\left[T^k < S^* \,\middle|\, \mathcal{F}_{t_i}\right] > 0\} = \{\mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k} < \xi_{t_i}\}$.
   Indeed, it follows from $t_{i+1} \leq T^k \leq S^*$ and the definition of S^* that $\{T^k < S^*\} = \{\xi_{T^k} < \xi_{t_i}\}$ up to a $\mathbb{P}$–null set. Hence,
   $$\mathbb{P}\left[T^k < S^* \,\middle|\, \mathcal{F}_{t_i}\right] = \mathbb{P}\left[\xi_{T^k} < \xi_{t_i} \,|\, \mathcal{F}_{t_i}\right] \quad \mathbb{P}\text{–a.s.}$$
   Up to a $\mathbb{P}$–null set, the latter conditional probability is strictly positive if and only if $\mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k} < \xi_{t_i}$. This proves claim (i).
2. *Up to a $\mathbb{P}$–null set we have* $\{\mathbb{P}\left[T^k < S^* \,\middle|\, \mathcal{F}_{t_i}\right] > 0\} \supset \{\mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k} < \Xi_{t_i,T^k}\}$.
   Sine $T^k \leq S^*$ $\mathbb{P}$–a.s. we have that
   $$\begin{aligned}\{\mathbb{P}\left[T^k < S^* \,\middle|\, \mathcal{F}_{t_i}\right] = 0\}&\\ &= \{\mathbb{P}\left[T^k = S^* \,\middle|\, \mathcal{F}_{t_i}\right] = 1\}\\ &\subset \{\mathbb{P}\left[\xi_{T^k} = \xi_{S^*} \,|\, \mathcal{F}_{t_i}\right] = 1\}\\ &\subset \{\mathbb{P}\left[\xi_{T^k} \geq \Xi_{t_i,S^*} \,|\, \mathcal{F}_{t_i}\right] = 1\}\end{aligned}$$

up to a $\mathbb{P}$–null set, where the last inclusion holds true since

$$\xi_{S^*} \geq \xi_{t_i} = \Xi_{t_i,S^*} \quad \mathbb{P}\text{–a.s.}$$

by definition of S^* and (40). Hence, up to a $\mathbb{P}$–null set we can write:

$$\begin{aligned} &\{\mathbb{P}\left[T^k < S^* \,\middle|\, \mathcal{F}_{t_i}\right] = 0\} \\ &\qquad = \{\mathbb{P}\left[\xi_{T^k} \geq \Xi_{t_i,S^*} \,|\, \mathcal{F}_{t_i}\right] = 1\} \cap \{\mathbb{P}\left[T^k = S^* \,\middle|\, \mathcal{F}_{t_i}\right] = 1\} \\ &\qquad \subset \{\mathbb{P}\left[\xi_{T^k} \geq \Xi_{t_i,T^k} \,\middle|\, \mathcal{F}_{t_i}\right] = 1\} \end{aligned}$$

where the last inclusion is true as

$$\{\mathbb{P}\left[T^k = S^* \,\middle|\, \mathcal{F}_{t_i}\right] = 1\} \subset \{\Xi_{t_i,S^*} = \Xi_{t_i,T^k}\} \in \mathcal{F}_{t_i}$$

by definition of $\Xi_{t_i,\cdot}$.
Passing to complements, the above inclusions imply

$$\begin{aligned} \{\mathbb{P}\left[T^k < S^* \,\middle|\, \mathcal{F}_{t_i}\right] > 0\} &\supset \{\mathbb{P}\left[\xi_{T^k} \geq \Xi_{t_i,T^k} \,\middle|\, \mathcal{F}_{t_i}\right] < 1\} \\ &= \{\mathbb{P}\left[\xi_{T^k} < \Xi_{t_i,T^k} \,\middle|\, \mathcal{F}_{t_i}\right] > 0\} \\ &= \{\mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k} < \Xi_{t_i,T^k}\} \end{aligned}$$

where the last equality follows from the $\mathcal{F}_{t_i}$–measurability of Ξ_{t_i,T^k} and the definition of $\mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k}$.

In order to complete the proof of (43) we use (42), (ii), and (i) to obtain that up to a $\mathbb{P}$–null set we have

$$\begin{aligned} \{T^k < T^{k+1}\} &= \{\xi_{T^k} = \mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k}\} \cap \{\mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k} < \Xi_{t_i,T^k}\} \\ &\subset \{\xi_{T^k} = \mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k}\} \cap \{\mathbb{P}\left[T^k < S^* \,\middle|\, \mathcal{F}_{t_i}\right] > 0\} \\ &= \{\xi_{T^k} = \mathcal{F}_{t_i}\text{–}\operatorname{ess\,inf} \xi_{T^k} < \xi_{t_i}\} \\ &\subset \{T^k < S^*\}, \end{aligned}$$

using $T^k \leq S^*$ $\mathbb{P}$–a.s. and the definition of S^* for the last inclusion. □

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Modeling Anticipations on Financial Markets

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Summary. The aim of the present survey is to give an outline of the modern mathematical tools which can be used on a financial market by a ”small” investor who possesses some information on the price process.

Financial markets obviously have asymmetry of information. That is, there are different types of traders whose behavior is induced by different types of information that they possess. Let us consider a ”small” investor who trades in an arbitrage free financial market so as to maximize the expected utility of his wealth at a given time horizon. We assume that he is in the following position: He possesses extra information about some functional Y of the future prices of a stock (e.g. value of the price at a given date, hitting times of given values, ...). Our basic question is then: What is the value of this information ? We can imagine two modeling approaches:

1. A strong approach: The investor knows the functional ω by ω. This modeling of the additional information was initiated in [41] using initial enlargement of filtration, a theory developed in [27], [28], and [29].
2. A weak approach: The investor knows the law of the functional Y under the effective probability of the market assumed to be unknown. This notion of weak information is defined in [5], [6], and further studied in [8].

In this chapter, we present and compare these two approaches.

1 Mathematical Framework

Let $T > 0$ be a constant finite time horizon. In what follows, we will work on a continuous arbitrage free financial market. Namely, we work on a filtered probability

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space $\left(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$ which satisfies the usual conditions (i.e. $\mathcal{F}$ is complete and right-continuous and $\mathcal{F}$ is assumed to be trivial). We assume that the price process of a given contingent claim is a continuous $d-$dimensional and $\mathcal{F}$-adapted square integrable local martingale $(S_t)_{0\leq t\leq T}$. In addition, we shall assume that the quadratic covariation matrix of the $d-$dimensional process S which is denoted by $\langle S\rangle$:

$$\langle S\rangle_t = \left(\langle S^i, S^j\rangle_t\right)_{1\leq i,j\leq d}$$

is almost surely valued in the space of positive matrix, which means that S is non-degenerate. For a matrix M, M^* will denote the transpose of M and for a vector $v \in \mathbb{R}^d$, $\operatorname{diag}(v)$ denotes the $d \times d$ matrix

$$\begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & v_d \end{pmatrix}.$$

Of course, since S is a local martingale under $\mathbb{P}$, the market just defined has no arbitrage (precisely, there are no arbitrage opportunities with tame portfolios, see Corollary 2 in [36]; see also [15] for a general view on the absence of arbitrage property). Moreover, to take a null drift under $\mathbb{P}$, means that we consider discounted prices (or equivalently, prices expressed in the bond numéraire). The key point is that we start from the *observable* dynamics of S (a model of S under $\mathbb{P}$ can be calibrated to market data) and not from the “true” but unobservable dynamics of S under a so-called “historical” measure. Our modeling of anticipations on S will provide both in the strong and the weak approach an additional drift which will be calibrated over the information that the insider possesses. The quadratic variation process, or volatility, will be not be affected by anticipations on S; indeed, from a mathematical point of view this quadratic variation is invariant both by enlargement of filtration and equivalent changes of probability. Moreover, it is important to note that we could actually start from any semimartingale model for the price process.

Definition 1. *The space $\mathcal{M}(S)$ of martingale measures is the set of probabilities $\widetilde{\mathbb{P}} \sim \mathbb{P}$ such that $(S_t)_{0\leq t\leq T}$ is an $\mathcal{F}$-adapted local martingale under $\widetilde{\mathbb{P}}$.*

Let us now precise what we mean by arbitrage on the financial market

$$\left(\Omega, (\mathcal{H}_t)_{0\leq t<T}, (S_t)_{0\leq t<T}, \mathbb{Q}\right)$$

where $\mathcal{H}$ is a filtration (right-continuous and $\mathbb{P}-$complete) which contains the natural filtration of S, and $\mathbb{Q}$ a probability measure equivalent to $\mathbb{P}$.

Definition 2. *We say that there is an arbitrage on the financial market*

$$\left(\Omega, (\mathcal{H}_t)_{0\leq t<T}, (S_t)_{0\leq t<T}, \mathbb{Q}\right),$$

if there exists a probability measure $\widetilde{\mathbb{Q}}$ *equivalent to* $\mathbb{Q}$ *such that* $(S_t)_{0\leq t<T}$ *is an* $\mathcal{H}$*-adapted local martingale under* $\widetilde{\mathbb{Q}}$.

Definition 3. *The space* $\mathcal{A}_{\mathcal{F}}(S)$ *of admissible strategies is the space of* $\mathbb{R}^d-$*valued and* $\mathcal{F}-$*predictable processes* Θ *integrable with respect to the price process* S*, such that*

$$\left(\int_0^t \Theta_u \cdot dS_u\right)_{0\leq t\leq T}$$

is a $\left(\widetilde{\mathbb{P}}, \mathcal{F}\right)$ *martingale for all* $\widetilde{\mathbb{P}}\in\mathcal{M}(S)$.

Remark 1. Θ_t^i represents the number of shares of the risky asset S^i held by an investor at time t and the wealth process associated with the strategy $\Theta \in \mathcal{A}(S)$ with initial capital x is given by

$$V_t = x + \int_0^t \Theta_u \cdot dS_u.$$

In particular, our strategies are self-financing.

Remark 2. This set of admissible strategies is restrictive. We used it because of the Definition 6 of admissible strategies in an enlarged filtration. Nevertheless, we believe that is possible to extend most of the presented results in a slightly much general setting.

We shall also very often assume that the financial market

$$\left(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}, (S_t)_{0\leq t\leq T}\right)$$

is complete in the sense that the martingale $(S_t)_{0\leq t\leq T}$ enjoys the following predictable representation property (in abbreviate PRP): For each $\mathcal{F}$-adapted local martingale $(M_t)_{0\leq t\leq T}$ there exists a predictable process Θ locally in L^2 such that

$$M_t = M_0 + \int_0^t \Theta_u \cdot dS_u, \quad t \leq T.$$

Remark 3. Under the previous assumption, we have

$$\mathcal{M}(S) = \{\mathbb{P}\}.$$

Indeed, if $\widetilde{\mathbb{P}} = D\ \mathbb{P} \in\mathcal{M}(S)$, then since $(S_t)_{0\leq t\leq T}$ is a local martingale under $\widetilde{\mathbb{P}}$, thanks to Girsanov's theorem we get that for all $1 \leq i \leq d$

$$d\langle S^i, D\rangle_t = 0.$$

But, by assumption, there exists Θ, locally in L^2, such that

$$D = 1 + \int_0^T \Theta_u \cdot dS_u$$

which implies that for all $1 \leq i \leq d$

$$\Theta_u^i = 0,\ 0 \leq u \leq T,$$

and hence

$$D = 1.$$

In what follows, we use the following notion of utility functions (see [32]):

Definition 4. *A utility function is a strictly increasing, strictly concave and twice continuously differentiable function*

$$U : (0, +\infty) \to \mathbb{R}$$

which satisfies

$$\lim_{x \to +\infty} U'(x) = 0, \quad \lim_{x \to 0^+} U'(x) = +\infty.$$

We use the convention that $U(x) = -\infty$ for $x \leq 0$. We shall denote by I the inverse of U', and by $\tilde{U}$ the convex conjugate of U:

$$\tilde{U}(y) = \max_{x>0} (U(x) - xy)$$

(that is, the Fenchel-Legendre transform of $-U(-x)$).

For a sake of simplicity, we limit ourselves to smooth utility functions, although it is also possible to obtain some results in the non-smooth case (see e.g. [10]). In our examples, we study the cases $U(x) = \ln(x)$ and $U(x) = x^\alpha$. The case $U(x) = e^{\alpha x}$ is also interesting; it does not fit into the previous definition, though our results remain true in this case.

Let us now introduce the object on which anticipations will be made.

Let $\mathcal{P}$ be a Polish space (for example $\mathcal{P} = \mathbb{R}^n$, $\mathcal{P} = C(\mathbb{R}_+, \mathbb{R}^n)$, etc...) endowed with its Borel $\sigma-$algebra $\mathcal{B}(\mathcal{P})$ and let $Y : \Omega \to \mathcal{P}$ be an $\mathcal{F}_T-$measurable random variable (it will be a functional of the trajectories of the price process). We denote by $\mathbb{P}_Y$ the law of Y and assume that Y admits a regular disintegration with respect to the filtration $\mathcal{F}$, precisely we assume that:

Assumption 1 *There exists a jointly measurable continuous in t and $\mathcal{F}-$adapted process*

$$\eta_t^y, \quad 0 \leq t < T, \quad y \in \mathcal{P},$$

satisfying for $dt \otimes \mathbb{P}_Y$ almost every $0 \leq t < T$ and $y \in \mathcal{P}$,

$$\mathbb{P}(Y \in dy \mid \mathcal{F}_t) = \eta_t^y\, \mathbb{P}_Y(dy). \tag{1}$$

This is a classical assumption of the theory of the initial enlargement of the filtration $\mathcal{F}$ by Y (see [27]) . This assumption is not very restrictive, and will be satisfied for *nice* functionals as it will be seen. The existence of a conditional density $\frac{\mathbb{P}(Y \in dy | \mathcal{F}_t)}{\mathbb{P}_Y(dy)}$ is the main point, the existence of a regular version follows from general results on stochastic processes (see [45]).

Remark 4. We can note here that, if we denote by $\mathbb{P}^y$ the disintegrated probability measure defined by $\mathbb{P}^y = \mathbb{P}(\cdot \mid Y = y)$, then the above assumption implies that for $t < T$,

$$\mathbb{P}^y_{/\mathcal{F}_t} = \eta^y_t \mathbb{P}_{/\mathcal{F}_t}.$$

In particular, for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, the process $(\eta^y_t)_{0 \le t < T}$ is a martingale in the filtration $\mathcal{F}$ (not uniformly integrable, as it is nicely illustrated in [23]).

Finally, we shall denote by $\mathcal{G}$ the filtration $\mathcal{F}$ initially enlarged with Y, i.e. $\mathcal{G}_t$ is the $\mathbb{P}-$completion of $\bigcap_{\varepsilon > 0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(Y))$, $t < T$.

2 Strong Information Modeling

The theory of initial enlargement of filtration has been developed by the French school of probability during the eighties (see [27], [28], [29], [47] and [46]). This theory has many deep applications, most of which have been worked out by T. Jeulin and M. Yor. In the past few years we have seen new interest in this theory because of its applications in mathematical finance in the topic of the asymmetry of information. Papers where applications of the enlargement of filtration technique is applied to portfolio optimization of an insider include [2], [3], [18], [20], [24], [25], and [41]. In this chapter, we review the theory of initial enlargement of filtration and its applications in finance. We shall **always** assume that the financial market

$$\left(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P}, (S_t)_{0 \le t \le T}\right)$$

is complete (i.e. that S enjoys the PRP, see section 1).

2.1 Some Results on Initial Enlargement of Filtration

Martingales of the Enlarged Filtration

Let us start this subsection with some simple remarks about martingales in the initially enlarged filtration $\mathcal{G}$.

Proposition 2.

- *Let $(M_t)_{0 \le t < T}$ be an $\mathcal{F}-$adapted process which is a (local) martingale in the filtration $\mathcal{G}$, then it is also a (local) martingale in the filtration $\mathcal{F}$.*
- *Let $(M_t)_{0 \le t < T}$ be an integrable process adapted to the filtration $\mathcal{G}$, then the two following statements are equivalent:*
  1. *For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, the process $(\mathbb{E}^y(M_t \mid \mathcal{F}_t))_{0 \le t < T}$ is a $\mathbb{P}^y-$(local) martingale in the filtration $\mathcal{F}$, $\mathbb{P}^y$ being the disintegrated probability measure defined by $\mathbb{P}^y = \mathbb{P}(\cdot \mid Y = y)$ and $\mathbb{E}^y$ the expectation under this measure.*
  2. *The process $(M_t)_{0 \le t < T}$ is a (local) martingale in the filtration $\mathcal{G}$.*

Proof. We make the proof of this proposition with true martingales because the case of local martingales is easily deduced by localization. The first point is trivial, indeed for $s < t < T$

$$\mathbb{E}\left(M_t \mid \mathcal{G}_s\right) = M_s$$

which implies, by conditioning with respect to $\mathcal{F}_s$,

$$\mathbb{E}\left(M_t \mid \mathcal{F}_s\right) = M_s.$$

Let us now prove the second point. Since for $s < t < T$, $A \in \mathcal{F}_s$ and $\Lambda \in \mathcal{B}\left(\mathcal{P}\right)$ we have

$$\mathbb{E}\left(\left(M_t - M_s\right) 1_{A \cap (Y \in \Lambda)}\right) = \int_\Lambda \mathbb{E}^y\left(\left(M_t - M_s\right) 1_A\right) \mathbb{P}_Y\left(dy\right)$$

it is easily seen by a monotone class theorem that our equivalence takes place. □

Of course, (local) martingales in the filtration $\mathcal{F}$ do not remain (local) martingales in the enlarged filtration, but as shown in the following section, they remain semimartingales.

Jacod's Theorem

Jacod's celebrated theorem says that a semimartingale which is adapted to the filtration $\mathcal{F}$ remains a semimartingale in the enlarged filtration $\mathcal{G}$. Before we state it, we start with a representation lemma which allows to give explicitly the semimartingale decomposition of S in the enlarged filtration. In what follows, $\mathcal{P}(\mathcal{F})$ denotes the predictable σ-field associated with the filtration $\mathcal{F}$.

Lemma 1. *(See [27]) There exists a $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathcal{P})$ measurable process*

$$\begin{array}{l} [0,T[\times \Omega \times \mathcal{P} \to \mathbb{R}^d \\ (\,t\quad,\ \omega\ ,\ \ y\,) \to \alpha_t^y\left(\omega\right) \end{array}$$

such that:

1. *For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$ and for $0 \le t < T$, $1 \le i \le d$,*

$$\mathbb{P}\left(\int_0^t \left(\alpha_u^y\right)^* d\langle S\rangle_u \alpha_u^y < +\infty\right) = 1$$

2. *For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$ and for $0 \le t < T$, $1 \le i \le d$,*

$$\langle \eta^y, S^i\rangle_t = \int_0^t \eta_u^y\left(\sum_{i=1}^d \alpha_u^{y,j} d\langle S^i, S^j\rangle_u\right).$$

Remark 5. We can choose α such that for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$ and for $0 \le t < T$,

$$\eta_t^y = \exp\left(\int_0^t \alpha_u^y \cdot dS_u - \frac{1}{2}\int_0^t \left(\alpha_u^y\right)^* d\langle S\rangle_u \alpha_u^y\right) \text{ on } \{\eta_t^y > 0\}.$$

We shall need (actually, only in the next section on the modeling of a weak anticipation; without this assumption Jacod's theorem remains true) the following additional assumption which ensures the existence of a version of α which allows to use filtering theory:

Assumption 3 *For almost every $t \in [0, T)$, we have $\mathbb{P}$-almost surely*

$$\int_0^t d\langle S^i, S^j \rangle_u \mathbb{E}\left(\left\| \alpha_u^Y \right\| \mid \mathcal{F}_u \right)^2 < +\infty, \quad 1 \leq i, j \leq d.$$

Theorem 1. *(See [27]) Under the probability $\mathbb{P}$ the price process $(S_t)_{0 \leq t < T}$ is a semimartingale in the filtration $\mathcal{G}$, and its decomposition is given by*

$$S_t = s_0 + \int_0^t d\langle S \rangle_u \alpha_u^Y + M_t, \qquad 0 \leq t < T \tag{2}$$

where $(M_t)_{0 \leq t < T}$ is a local martingale in the filtration $\mathcal{G}$ such that

$$\langle S \rangle = \langle M \rangle.$$

Proof. From Proposition 2, it is enough to show that for $a.e.\ y \in \mathcal{P}$, the process

$$M_t^y := -\int_0^t d\langle S \rangle_u \alpha_u^y + S_t$$

is a $\mathbb{P}^y$−local martingale in the filtration $\mathcal{F}$. But this is a direct consequence of Girsanov's theorem, because $(\eta_t^y)_{0 \leq t < T}$ is the density process of $\mathbb{P}^y$ with respect to $\mathbb{P}$ (see Remark 4). □

Hence the class of semimartingales is preserved by an initial enlargement of filtration. Related to this is Stricker's theorem (see [44]): If a process is a semimartingale in an enlarged filtration, then it is a semimartingale in its own filtration.

We conclude this paragraph with a converse of Jacod's theorem. As it will be seen later, this theorem makes the link between the strong and the weak approach and also shows that the natural filtration of M (as defined in the previous Theorem) is strictly included in $\mathcal{G}$.

Theorem 2. *(See [5]) Let $\mathbb{Q}$ be a probability measure on Ω which is equivalent to $\mathbb{P}$ with a bounded density. If the process $(M_t)_{0 \leq t < T}$ defined by*

$$M_t := S_t - \int_0^t d\langle S \rangle_u \alpha_u^Y, \qquad 0 \leq t < T$$

is a local martingale under $\mathbb{Q}$ in the filtration $\mathcal{G}$, then there exists a probability measure ν on $\mathcal{P}$ such that

$$\mathbb{Q} = \int_{\mathcal{P}} \mathbb{P}\left(\cdot \mid Y = y \right) \nu\left(dy \right).$$

Proof. For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, let us denote by $\mathbb{Q}^y$ the conditional probability $\mathbb{Q}(\cdot \mid Y = y)$. From our assumption and Proposition 2, the process M is, under $\mathbb{Q}^y$, a local martingale. Now, because $\mathbb{Q}$ is assumed to be equivalent to $\mathbb{P}$, it is easily seen that for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, $\mathbb{Q}^y$ is locally absolutely continuous on $\mathcal{F}$ with respect to $\mathbb{P}$. Hence, by Girsanov's theorem, for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$,

$$d\mathbb{Q}^y_{/\mathcal{F}_t} = \eta^y_t d\mathbb{P}_{/\mathcal{F}_t} \ , t < T.$$

Since we also have, for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$,

$$d\mathbb{P}^y_{/\mathcal{F}_t} = \eta^y_t d\mathbb{P}_{/\mathcal{F}_t} \ , t < T,$$

as explained in Remark 4, where $\mathbb{P}^y$ is the conditional probability $\mathbb{P}(\cdot \mid Y = y)$, we immediately deduce

$$\mathbb{Q}^y = \mathbb{P}^y$$

for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, and hence

$$\mathbb{Q} = \int_{\mathcal{P}} \mathbb{P}(\cdot \mid Y = y)\, \nu(dy)$$

where ν is the law of Y under $\mathbb{Q}$. □

Martingale Preserving Measure and PRP in the Enlarged Filtration

In all this paragraph, we shall assume that for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, the process $(\eta^y_t)_{0 \le t < T}$ is strictly positive $\mathbb{P}$-a.s.

Lemma 2. *(See [3]) The process*

$$Z^Y_t := \frac{1}{\eta^Y_t},\ t < T$$

is a $\mathbb{P}$-martingale (not necessarily uniformly integrable) in the enlarged filtration $\mathcal{G}$. Moreover, it satisfies

$$\mathbb{E}\left(Z^Y_t \mid \mathcal{F}_t\right) = 1,\ t < T. \tag{3}$$

Proof. For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, the process $(Z^y_t)_{0 \le t < T}$ is a $\mathbb{P}^y-$martingale in the filtration $\mathcal{F}$ because it is the density process of $\mathbb{P}^y$ with respect to $\mathbb{P}$. We conclude with Proposition 2. Now, for the second point, we claim that

$$\mathbb{E}\left(Z^Y_t \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} Z^y_t \eta^y_t \mathbb{P}_Y(dy) = 1,$$

which completes the proof. □

Definition 5. *The probability measure $\widetilde{\mathbb{P}}_t$, $t < T$, defined on $\mathcal{G}_t$ by*

$$\widetilde{\mathbb{P}}_t = Z^Y_t\ \mathbb{P}_{/\mathcal{G}_t}$$

is called the $[0,t]$ $-$martingale preserving measure associated with $\mathbb{P}$.

Remark 6. As a consequence of (3), we can note that for $t < T$ the law of an $\mathcal{F}-$adapted process $(X_s)_{0\leq s\leq t}$ is the same under $\mathbb{P}_{/\mathcal{F}_t}$ as under $\widetilde{\mathbb{P}}_t$.

Remark 7. We have the following representation result

$$Z_t^Y = \exp\left(-\int_0^t \alpha_u^Y \cdot dM_u - \frac{1}{2}\int_0^t \left(\alpha_u^Y\right)^* d\langle M\rangle_u \alpha_u^Y\right),\ t < T.$$

The terminology of martingale preserving measure stems from the following theorem.

Theorem 3. *(See [3]) For $t < T$, any $\mathbb{P}-$(local) martingale adapted to $(\mathcal{F}_s)_{0\leq s\leq t}$ is also a $\widetilde{\mathbb{P}}_t-$(local) martingale in the enlarged filtration $(\mathcal{G}_s)_{0\leq s\leq t}$and thus a $\widetilde{\mathbb{P}}_t-$(local) martingale in the filtration $(\mathcal{F}_s)_{0\leq s\leq t}$.*

which is a consequence of the following easy lemma.

Lemma 3. *(See [3]) For $t < T$, under the probability $\widetilde{\mathbb{P}}_t$, the $\sigma-$algebras $\mathcal{F}_t$ and $\sigma(Y)$ are independent.*

We conclude this subsection on the general theory of initial enlargement of a filtration with a particular case of the PRP of the martingale preserving measure in the enlarged filtration obtained by Amendinger [1]. This representation result is an easy consequence of the stability of the PRP by Girsanov's transforms (see Proposition 17.1. in [47]).

Theorem 4. *Let $t < T$. For any $\widetilde{\mathbb{P}}_t-$(local) martingale $(M_s)_{0\leq s\leq t}$ adapted to $(\mathcal{G}_s)_{0\leq s\leq t}$ and which satisfies $M_0 = 0$, there exists a $\mathcal{G}-$predictable process Θ which is integrable with respect to S and which satisfies*

$$M_s = \int_0^s \Theta_u dS_u,\ s\leq t.$$

2.2 Examples of Initial Enlargement of Filtration

We present now some illustrative examples of enlargement of filtration. For further examples, we refer the interested reader to the "Récapitulatif" in [29], pp. 305-313 and Chapter 12 of [47].

Stochastic Analysis Warm Up

Here, we use stochastic analysis to obtain explicit computations for the enlargement formula of the natural filtration of the coordinate process. First, we recall some basic definitions of stochastic analysis (for further details we refer to the book [40]). Consider the $d-$dimensional Wiener space of continuous paths

$$\mathbb{W}^d = \left(\mathcal{C}_T^d, (\mathcal{F}_t)_{0\leq t\leq T}, (X_t)_{0\leq t\leq T}, \mathbb{P}\right)$$

where:

1. $\mathcal{C}_T^d$ is the space of continuous functions $f:[0,T]\to\mathbb{R}^d$, such that $f(0)=0$
2. $(X_t)_{0\le t\le T}$ is the coordinate process defined by $X_t(f)=f(t)$
3. $(\mathcal{F}_t)_{t\ge 0}$ is the natural filtration of $(X_t)_{0\le t\le T}$
4. $\mathbb{P}$ is the Wiener measure.

We assume furthermore that

$$\mathcal{P}=\mathbb{R}^N$$

for some integer $N\ge 1$ and that Y belongs to $\left(\mathbb{D}^{1,2}\right)^N$. We recall (see [40] pp. 26) that the Hilbert space $\mathbb{D}^{1,2}$ is the closure of the class of smooth cylindrical random variables with respect to the norm

$$\|F\|_{1,2}=\left(\mathbb{E}\left(F^2\right)+\mathbb{E}\left(\|\mathbf{D}F\|_{L^2}^2\right)\right)^{\frac{1}{2}}$$

where $\mathbf{D}$ is the gradient defined for a functional $F=\Phi\left(X\left(f_1\right),...,X\left(f_d\right)\right)$ by

$$\mathbf{D}F=\left(Jac\phi\right)\left(f_1,...,f_d\right)$$

where $\Phi:\mathbb{R}^d\to\mathbb{R}^d$ is a smooth function whose Jacobian matrix is denoted $Jac\phi$, and $(f_i)_{i=1,\dots,d}\in\left(L^2\right)^d$. We give now in this setting, by means of the Clark-Ocone formula, some expressions for α and η in the case where $S_t=X_t$, $t\ge 0$.

Theorem 5. *(see [5]) Assume that for almost every $t<T$,*

$$\int_{\mathbb{R}^N}\left|\mathbb{E}\left(e^{i\lambda\cdot Y}\mid\mathcal{F}_t\right)\right|d\lambda<+\infty \tag{4}$$

then Y has a density p with respect to the Lebesgue measure which is given by

$$p(y)=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{-iy\cdot\lambda}\,\mathbb{E}\left(e^{i\lambda\cdot Y}\right)d\lambda,$$

and for $\mathbb{P}_Y-a.e.\ y\in\mathbb{R}^N$ and for almost every $t<T$,

$$p(y)\,\eta_t^y=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{-iy\cdot\lambda}\,\mathbb{E}\left(e^{i\lambda\cdot Y}\mid\mathcal{F}_t\right)d\lambda. \tag{5}$$

Moreover, if for $\mathbb{P}_Y-a.e\ y\in\mathbb{R}^N$ and for almost every $t<T$,

$$\int_{\mathbb{R}^N}\left\|\mathbf{D}_t\mathbb{E}\left(e^{i\lambda\cdot Y}\mid\mathcal{F}_t\right)\right\|d\lambda<+\infty,\ p(y)\neq 0,$$

then for $\mathbb{P}_Y-a.e\ y\in\mathbb{R}^N$ and for almost every $t<T$, $\eta_t^y\in Dom\left(\mathbf{D}\right)$ and

$$\mathbf{D}_t\eta_t^y=\eta_t^y\alpha_t^y. \tag{6}$$

Proof. Let m be a function on $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} |m(y)|\, dy < +\infty$$

Let now $\widetilde{m}$ be the Fourier transform of m defined on $\mathbb{R}^N$ by

$$\widetilde{m}\,(y) = \int_{\mathbb{R}^N} e^{iy\cdot\lambda} m\,(d\lambda)\,.$$

We have for $t < T$,

$$\int_{\mathbb{R}^N} \eta_t^y\; \widetilde{m}\,(y)\, \mathbb{P}_Y\,(dy) = \mathbb{E}\,(\widetilde{m}\,(Y) \mid \mathcal{F}_t)\,.$$

But,

$$\mathbb{E}\,(\widetilde{m}\,(Y) \mid \mathcal{F}_t) = \int_{\mathbb{R}^N} \mathbb{E}\left(e^{iY\cdot\lambda} \mid \mathcal{F}_t\right) m\,(\lambda)\, d\lambda$$

hence,

$$\int_{\mathbb{R}^N} \eta_t^y\; \widetilde{m}\,(y)\, \mathbb{P}_Y\,(dy) = \int_{\mathbb{R}^N} \mathbb{E}\left(e^{iY\cdot\lambda} \mid \mathcal{F}_t\right) m\,(\lambda)\, d\lambda.$$

Since the previous equality takes place for all m, this implies, thanks to the inversion formula for the Fourier transform:

$$m\,(\lambda) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-iy\cdot\lambda} \widetilde{m}\,(y)\, dy$$

that for $\mathbb{P}_Y - a.e\; y \in \mathbb{R}^N$ and for all $t < T$,

$$\eta_t^y\; \mathbb{P}_Y(dy) = \frac{1}{(2\pi)^N} \left(\int_{\mathbb{R}^N} e^{-iy\cdot\lambda}\; \mathbb{E}\left(e^{i\lambda\cdot Y} \mid \mathcal{F}_t\right) d\lambda \right) dy.$$

This provides easily the first part of our theorem.

Assume now that for $\mathbb{P}_Y - a.e.$, $y \in \mathbb{R}^N$ and for $a.e.\; t < T$,

$$\int_{\mathbb{R}^N} \left\| \mathbf{D}_t \mathbb{E}\left(e^{i\lambda\cdot Y} \mid \mathcal{F}_t\right) \right\| d\lambda < +\infty.$$

Because for $\mathbb{P}_Y - a.e.$, $y \in \mathbb{R}^N$ and for $a.e.\; t < T$, $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left(e^{i\lambda\cdot Y} \mid \mathcal{F}_t\right) \in \mathbb{D}^{1,2}$$

it is easily seen that $\eta_t^y \in Dom\,(\mathbf{D})$. Moreover, let us consider a probability measure μ on $\mathbb{R}^N$ such that μ is equivalent to $\mathbb{P}_Y$ and $\xi := \frac{d\mu}{d\mathbb{P}_Y}$ admits a bounded continuously differentiable version. From the Clark-Ocone formula, we have:

$$d\left[\int_{\mathbb{R}^N} \eta_t^y \mu\,(dy)\right] = \mathbb{E}\,(\mathbf{D}_t \xi\,(Y) \mid \mathcal{F}_t) \cdot dX_t$$

which implies,

$$\int_{\mathbb{R}^N} \alpha_t^y \eta_t^y \mu(dy) = \mathbf{D}_t \mathbb{E}(\xi(Y) \mid \mathcal{F}_t) = \mathbf{D}_t \int_{\mathbb{R}^N} \eta_t^y \mu(dy)$$

for a.e. $t \in [0,T]$ and the conclusion follows easily because μ was arbitrary. □

Remark 8. Of course, formula (5) remains true under the assumption (4) even if $Y \notin \left(\mathbb{D}^{1,2}\right)^N$.

Remark 9. The formula 6 can be found in [26] (Proposition A.1.) in an equivalent form. Indeed, in this paper the authors have developed a Malliavin Calculus for measure valued random variables and gave a sense to the following formula

$$\alpha_t^y = \frac{\mathbb{E}(\mathbf{D}_t \delta_Y \mid \mathcal{F}_t)(dy)}{\mathbb{P}(Y \in dy \mid \mathcal{F}_t)}.$$

Initial Enlargement with the Terminal Value of a Diffusion

In this subsection, we consider the case $\mathcal{P} = \mathbb{R}^d$ and $Y = S_T$. We assume furthermore that the dynamics of $(S_t)_{0 \le t \le T}$ under $\mathbb{P}$ are given by

$$S_t = s_0 + \int_0^t \operatorname{diag}(S_u)\, \sigma(S_u)\, dB_u, \qquad 0 \le t \le T \tag{7}$$

where $(B_t)_{0 \le t \le T}$ is a d-dimensional standard Brownian motion whose filtration is $\mathcal{F}$, $s_0 \in \left(\mathbb{R}_+^*\right)^d$ and σ a positive definite symmetric bounded C^∞ function with bounded partial derivatives function satisfying

$$\inf_{x \in \mathbb{R}^d} \|\sigma\sigma^*(x)\| \ge a > 0.$$

The last assumption implies (Hörmander's theorem, see [40] Chap. 2) the differentiability of the transition function p_t, $0 < t \le T$, defined by

$$p_t(x,y)\, dy = \mathbb{P}(S_t \in dy \mid S_0 = x).$$

Moreover, it is easily seen that we obviously are in the assumptions of the theory of the initial enlargement of $\mathcal{F}$ with S_T. Namely, for $0 \le t < T$,

$$\mathbb{P}(S_T \in dy \mid \mathcal{F}_t) = p_{T-t}(S_t, y)\, dy$$

which implies that for $\mathbb{P}_{S_T} - a.e.\ y \in \mathbb{R}^d$ and $t < T$

$$\eta_t^y = \frac{p_{T-t}(S_t, y)}{p_T(s_0, y)}. \tag{8}$$

In what follows, for a smooth function $f(x,y)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$, ∇f denotes the gradient computed with respect to the first variable.

Theorem 6. *In the filtration $\mathcal{G}$, the process $(S_t)_{0\leq t<T}$ admits the following semimartingale decomposition*

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \frac{\nabla p_{T-t}}{p_{T-t}}(S_u, S_T) + \int_0^t diag(S_u)\,\sigma(S_u)\,d\beta_u$$

where $(\beta_t)_{0\leq t\leq T}$ is a d-dimensional standard Brownian motion adapted to the filtration $\mathcal{G}$ (and hence independent of S_T).

Proof. In this case, from the Markov property of S (see (8), or [21] for further details), we have for $\mathbb{P}_{S_T} - a.e.\ y \in \mathbb{R}^d$

$$\eta_t^y = \frac{p_{T-t}(S_t, y)}{p_T(s_0, y)},\ t < T.$$

Hence, thanks to Itô's formula

$$\alpha_t^y = \frac{\nabla p_{T-t}}{p_{T-t}}(S_t, y),$$

which leads, according to Jacod's theorem, to the expected result. □

Initial Enlargement with the First Hitting Time of a Level of the Brownian Motion

In this paragraph, we give the formula for the enlargement with the first hitting time of $a > 0$ by a standard Brownian motion B. Let us denote by T_a this stopping time, i.e.

$$T_a = \inf_{t\geq 0}\{t,\ B_t = a\}.$$

In this case, the assumption of the existence of η is not satisfied. Nevertheless, as it is seen in the proof of Theorem 7, there exists a jointly measurable continuous in t and $\mathcal{F}_{\cdot\wedge T_a}-$adapted process

$$\eta_t^y,\quad 0 \leq t < T_a,\quad y \in \mathbb{R}_+^*$$

satisfying for $dt \otimes \mathbb{P}_{T_a}$ almost every $0 \leq t < T_a$ and $y \in \mathbb{R}_+^*$,

$$\mathbb{P}(T_a \in dy \mid \mathcal{F}_t,\ t < T_a) = \eta_t^y\, \mathbb{P}(T_a \in dy).$$

Moreover, a process α can be associated (up to time T_a) with η in the same way as in the Lemma 1. In [28], T. Jeulin found the following enlargement formula:

Theorem 7. *In the filtration $\mathcal{G} = \mathcal{F}_{\cdot\wedge T_a} \vee \sigma(T_a)$, the process B admits the following semimartingale decomposition*

$$B_t = \beta_t - \int_0^t \left(\frac{1}{a - B_s} - \frac{a - B_s}{T_a - s}\right) ds,\quad t < T_a \tag{9}$$

where β is a standard Brownian motion adapted to the filtration $\mathcal{G}$ (and hence independent of T_a).

Proof. It is easily shown, by using exponential martingales that for $\alpha \in \mathbb{R}$,

$$\mathbb{E}\left(e^{-\frac{\alpha^2}{2}T_a} \mid \mathcal{F}_t,\ t < T_a\right) = e^{\alpha(B_t - a) - \frac{\alpha^2}{2}t}.$$

It stems from this, by inverting the previous Laplace transform that for $y > 0$,

$$\frac{\mathbb{P}\left(T_a \in dy \mid \mathcal{F}_t,\ t < T_a\right)}{\mathbb{P}\left(T_a \in dy\right)} = \frac{a - B_t}{a}\left(\frac{y}{y-t}\right)^{3/2} e^{\frac{a^2}{2y} - \frac{(a-B_t)^2}{2(y-t)}}$$

which gives the expected result after straightforward computations. □

Initial Enlargement with the Perpetuity

We consider now the case $\mathcal{P} = \mathbb{R}$ and assume that the dynamics of S under $\mathbb{P}$ are one-dimensional and given by

$$S_t = \exp\left(B_t - \frac{1}{2}t\right),\ t \geq 0$$

where $(B_t)_{0 \leq t \leq T}$ is a one-dimensional standard Brownian motion whose filtration is $\mathcal{F}$. We recall that the functional

$$Y = \int_0^{+\infty} S_t^2 dt = \int_0^{+\infty} e^{2B_t - t} dt$$

is well defined and distributed, up to a multiplicative constant, as the inverse of a gamma law, precisely (see [19]):

$$\int_0^{+\infty} S_t^2 dt \sim \frac{1}{2\gamma_{1/2}},$$

which means that

$$\mathbb{P}\left(\int_0^{+\infty} S_t^2 dt \in dy\right) = \frac{1}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)} \frac{e^{-\frac{1}{2y}}}{y^{3/2}} 1_{y>0} dy.$$

We have then the following enlargement formula.

Theorem 8. *In the filtration $\mathcal{G} = \mathcal{F} \vee \sigma\left(\int_0^{+\infty} S_t^2 dt\right)$, the process S admits the following semimartingale decomposition*

$$S_t = 1 + \int_0^t S_u \left(1 - \frac{S_u^2}{\int_u^{+\infty} S_v^2 dv}\right) du + \int_0^t S_u d\beta_u\ ,\ t \geq 0 \tag{10}$$

where $(\beta_t)_{t \geq 0}$ is a standard Brownian motion adapted to the enlarged filtration (and hence independent of $\int_0^{+\infty} S_t^2 dt$).

Proof. From the Dubins-Schwarz theorem (which appears here as a special case of the Lamperti's relation, see [43] pp. 452), there exists a standard Brownian motion γ such that

$$S_t = 1 - \gamma_{\int_0^t S_u^2 du}.$$

Hence,

$$\int_0^{+\infty} S_t^2 dt = T_1$$

where

$$T_1 = \inf_{t \geq 0} \{t, \gamma_t = 1\}$$

and we conclude with the enlargement formula (9). □

Remark 10. The decomposition (10) implies (see [7])

$$S_t = \frac{e^{\beta_t + \frac{1}{2}t}}{1 + \frac{\int_0^t e^{2\beta_s + \mu s} ds}{\int_0^{+\infty} S_t^2 dt}}, t \geq 0.$$

2.3 Utility Maximization with Strong Information

The Financial Market of the Informed Insider

We now apply to finance the general results of initial enlargement of filtration presented in the previous sections. For this, we assume that for $a.e.\ y \in \mathcal{P}$, the process $(\eta_t^y)_{0 \leq t < T}$ is strictly positive $\mathbb{P}$-a.s. Hence, we can use the martingale preserving measure introduced earlier in Section 2.1. we study here the financial market

$$\left(\Omega, (\mathcal{G}_t)_{0 \leq t < T}, \mathbb{P}, (S_t)_{0 \leq t < T}\right) \tag{11}$$

and solve the portfolio optimization problem associated with this model. The first remark is the following (see Definition 2).

Proposition 4. *For $t < T$, there is no arbitrage in the time interval $[0, t]$. But there is an arbitrage in the time interval $[0, T]$.*

Proof. Indeed, from Theorem 3, the martingale preserving measure $\widetilde{\mathbb{P}}_t$ is a martingale measure for S. Now, let us assume that there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ which is equivalent to $\mathbb{P}$ and such that S is a martingale under $\mathbb{Q}$. In this case, on the time-interval $[0, t]$, S is a local martingale. It implies from Theorem 4 that $\mathbb{Q}_{/\mathcal{G}_t} = \widetilde{\mathbb{P}}_t$, which can not happen because the $\mathbb{P}-$martingale $\left(\frac{1}{\eta_t^Y}\right)_{0 \leq t < T}$ is not uniformly integrable. □

Value of the Strong Information

Let now U be a utility function as defined in Section 1.

Definition 6. *The space $\mathcal{A}_{\mathcal{G}}(S)$ of admissible strategies is the space of $\mathbb{R}^d-$valued and $\mathcal{G}-$predictable processes $(\Theta_s)_{s<T}$ integrable with respect to the price process S, such that*

$$\left(\frac{1}{\eta_t^Y}\int_0^t \Theta_u \cdot dS_u\right)_{0\le t<T}$$

is a $(\mathbb{P},\mathcal{G})$-martingale.

With this set of admissible strategies, we classically associate the following portfolio optimization problem.

Portfolio Optimization Problem on $[0,t]$: *For $t\in[0,T]$, the insider's portfolio optimization problem on $[0,t]$ is to find*

$$u(x,Y,t) := \sup_{\Theta\in\mathcal{A}_{\mathcal{G}}(S)} \mathbb{E}\left(U\left(x+\int_0^t \Theta_u dS_u\right)\right),$$

$x>0$ *being the initial endowment of the insider.*

We restrict ourselves to the time-interval $[0,t]$, because of the presence of the arbitrage discussed in the previous proposition and now solve the portfolio optimization problem on $[0,t]$.

Theorem 9. *(See [2]) Let $t\in[0,T)$, $x>0$ and let us assume that there exists an $\sigma(Y)-$measurable random variable $\Lambda_t(x):\Omega\to(0,+\infty)$ with*

$$\mathbb{E}\left(\frac{1}{\eta_t^Y} I\left(\frac{\Lambda_t(x)}{\eta_t^Y}\right) \mid Y\right) = x,$$

then

$$u(x,Y,t) = \mathbb{E}\left((U\circ I)\left(\frac{\Lambda_t(x)}{\eta_t^Y}\right)\right).$$

Proof. Let us set

$$V_t = I\left(\frac{\Lambda_t(x)}{\eta_t^Y}\right).$$

First, we note that, according to Theorem 4, there exist $\Theta\in\mathcal{A}_{\mathcal{G}}(S)$ such that

$$V_t = x + \int_0^t \Theta_u \cdot dS_u.$$

Since U is concave, we have

$$U(b) \ge U(a) + U'(b)(b-a),\ a,b\in(0,+\infty).$$

Hence

$$U(V_t) \geq U\left(\widetilde{V}_t\right) + \frac{\Lambda_t(x)}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right)$$

where

$$\widetilde{V}_t = x + \int_0^t \widetilde{\Theta}_u \cdot dS_u,$$

with

$$\widetilde{\Theta} \in \mathcal{A}_{\mathcal{G}}(S).$$

Even if $\frac{\Lambda_t(x)}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right)$ is not integrable, we can take generalized conditional expectations to obtain

$$\mathbb{E}\left(\frac{\Lambda_t(x)}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right) \mid Y\right) = \Lambda_t(x)\,\mathbb{E}\left(\frac{1}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right) \mid Y\right) = 0.$$

We conclude hence

$$\mathbb{E}\left(U(V_t) \mid Y\right) \geq \mathbb{E}\left(U\left(\widetilde{V}_t\right) \mid Y\right),$$

which gives the expected result. □

As an illustration of the previous theorem, we give the optimal expected utility in the case of the most commonly used utility functions.

Example 1. Let $\alpha \in (0,1)$ and $U(x) = \frac{x^\alpha}{\alpha}$, then

$$I(y) = y^{\frac{1}{\alpha-1}},$$

$$\Lambda_t(x) = \frac{x^{\alpha-1}}{\mathbb{E}\left(\left(\eta_t^Y\right)^{\frac{\gamma}{1-\gamma}} \mid Y\right)},$$

and

$$u(x,Y,t) = \frac{x^\alpha}{\alpha}\mathbb{E}\left(\mathbb{E}\left[\left(\eta_t^Y\right)^{\frac{\alpha}{1-\alpha}} \mid Y\right]^{1-\alpha}\right).$$

Example 2. Let $U(x) = \ln x$, then

$$I(y) = \frac{1}{y},$$

$$\Lambda_t(x) = \frac{1}{x},$$

and

$$u(x,Y,t) = \ln x + \mathbb{E}\left(\ln \eta_t^Y\right).$$

2.4 Comments

The fact that there always is an arbitrage in the financial market

$$\left(\Omega, (\mathcal{G}_t)_{0\leq t<T}, \mathbb{Q}, (S_t)_{0\leq t<T}\right)$$

is one of the principal problem of this approach. From the mathematical point of view, it is easily understood because the knowledge of a functional ω by ω is very restrictive. In Chapter 3, we develop the notion of weak information, which is much more flexible and which leaves more freedom on the model used by the informed insider.

Nevertheless, by a change of filtration, there is some possibilities to allow much freedom on the anticipation. For instance, we can enlarge the "public" filtration $\mathcal{F}$ by $Y+N$ where N is a noise independent of $\mathcal{F}$ and constant in the time. The computations and theorems associated with this kind of enlargement are studied in a very general setting in [2], and [1] for the PRP properties. Another possibility is to use a part of the theory of progressive enlargement of filtration (for further details on it we refer to [47] Section 12.2.). Roughly speaking, we would like that the "noise" N could evolve in the time. Typically, it is natural to study enlargement formulas associated with the filtration $\mathcal{F}_t \vee \sigma\left(Y + W_{T-s}, s \leq t\right)$ where W is a standard Brownian motion independent of $\mathcal{F}_T$. This idea is presented in [14], where the authors give, more generally, an enlargement formula associated with a filtration of the form $\mathcal{F}_t \vee \sigma\left(F(Y, W_s), s \leq t\right)$, W being this time any process independent of $\mathcal{F}_T$ and F a Borel function. In this setting, the authors proved that if the rate at which the additional noise in the insider's information vanishes is slow enough, then there is no arbitrage and the additional utility of the insider is finite. Let us see briefly more precisely what kind of "progressive" enlargement formulas can be obtained in full generality. Consider a filtration $\mathcal{W}$ independent of $\mathcal{F}_T$, and $\mathcal{H}$ a sub-filtration of $\mathcal{G} \vee \mathcal{W}$ which contains $\mathcal{F}$. We have then the following enlargement formula.

Theorem 10. *Assume that for almost every $t \in [0,T)$, we have $\mathbb{P}$-almost surely*

$$\int_0^t d\langle S^i, S^j\rangle_u \left\|\mathbb{E}\left(\alpha_u^Y \mid \mathcal{H}_u\right)\right\| < +\infty, \quad 1 \leq i,j \leq d$$

then, under the probability $\mathbb{P}$ the price process $(S_t)_{0\leq t<T}$ is a semimartingale in the filtration $\mathcal{H}$, and its decomposition is given by

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \mathbb{E}\left(\alpha_u^Y \mid \mathcal{H}_u\right) + M_t, \qquad 0 \leq t < T \tag{12}$$

where $(M_t)_{0\leq t<T}$ is a local martingale such that

$$\langle S\rangle = \langle M\rangle.$$

This theorem is easily understood by the "méthode des Laplaciens approchés" (see [16]). Namely, the bounded variation part A of a semimartingale N in the filtration $\mathcal{H}$ is

$$A_t = \lim_{h \to >0} \int_0^t \frac{\mathbb{E}\left(N_{s+h} - N_s \mid \mathcal{H}_s\right)}{h} ds$$

as soon as the right hand exists in L^1, and we know that S is semimartingale in the filtration $\mathcal{H}$ because it is in the filtration $\mathcal{G} \vee \mathcal{W}$ from Jacod's theorem. From a financial point of view, we can note that the financial market

$$\left(\Omega, \left(\mathcal{H}_t\right)_{t<T}, \left(S_t\right)_{t<T}, \mathbb{P}\right)$$

is not complete, contrary to the market

$$\left(\Omega, \left(\mathcal{G}_t\right)_{t<T}, \left(S_t\right)_{t<T}, \mathbb{P}\right).$$

Indeed, it is clear that there are many martingale measures for S. For instance,

$$\mathbb{Q}_{/\mathcal{H}_t} = \mathbb{E}\left(\frac{D_t}{\eta_t^Y} \mid \mathcal{H}_t\right) \mathbb{P}_{/\mathcal{H}_t}$$

is a martingale measure for S (in the filtration $\mathcal{H}$) as soon as D is a positive martingale adapted to the filtration $\mathcal{W}$ such that $\mathbb{E}(D) = 1$.

3 Weak Information Modeling

We now turn to the weak approach. The main difference with the strong one is that there is no change of filtration but only a change a probability. Nevertheless, as it will be seen, from a mathematical point of view, all the results relative to initial enlargement of filtration can be recover from the weak approach.

The topic of investors with additional weak information was initiated in [5] and [6] and further studied in [8]. We present these works in a continuous setting.

We keep the notations of Section 1 and 2 and consider here an insider who is only weakly informed on Y. This means that the he has knowledge of the filtration $\mathcal{F}$ and of the law of Y. More precisely, with Y we associate a probability measure ν on $\mathcal{P}$. We assume that ν is equivalent to $\mathbb{P}_Y$ with a bounded density. The probability ν should be interpreted as the law of Y under the effective probability of the market.

In this section, we shall assume that the financial market

$$\left(\Omega, \left(\mathcal{F}_t\right)_{0 \leq t \leq T}, \mathbb{P}, \left(S_t\right)_{0 \leq t \leq T}\right)$$

is complete, i.e. that the process S enjoys the PRP.

3.1 Conditioning of a Functional

Minimal Probability Associated with a Conditioning

We first associate with the weak information (Y, ν) a probability which will appear as canonical.

Definition 7. *The probability measure $\mathbb{P}^{\nu}$ defined on $(\Omega, \mathcal{F}_T)$ by:*

$$\mathbb{P}^{\nu}(A) = \int_{\mathcal{P}} \mathbb{P}(A \mid Y = y)\,\nu(dy), \; A \in \mathcal{F}_T$$

is called the minimal probability associated with the weak information (Y, ν).

Here are some immediate consequences of this definition:

1. If $F : \Omega \to \mathbb{R}$ is a bounded random variable then
$$\mathbb{E}^{\nu}(F \mid Y) = \mathbb{E}(F \mid Y)$$
2. The law of Y under $\mathbb{P}^{\nu}$ is ν
3. $\mathbb{P}^{\nu} = \mathbb{P} \Leftrightarrow \nu = \mathbb{P}_Y$
4. The following equivalence relationship takes place
$$d\mathbb{P}^{\nu} = \frac{d\nu}{d\mathbb{P}_Y}(Y)\; d\mathbb{P}$$
5. If $A \in \mathcal{F}_T$ is $\mathbb{P}-$independent of Y then it is also $\mathbb{P}^{\nu}-$independent of Y
6. If we can choose a version of the map $y \to \mathbb{P}(\cdot \mid Y = y)$ which is continuous in the topology of weak convergence of probability measures, then the map $\nu \to \mathbb{P}^{\nu}$ is also continuous in this topology.

In order to justify the word *minimal* assigned to $\mathbb{P}^{\nu}$ we consider a convex function

$$\varphi : \mathbb{R}_+ \to \mathbb{R}$$

and denote by $\mathcal{E}^{\nu}$ the set of probability measures on Ω which are equivalent to $\mathbb{P}$ and such that the law of Y under $\mathbb{Q}$ is ν.

Proposition 5. *(See [5]) We have*

$$\min_{\mathbb{Q} \in \mathcal{E}^{\nu}} \mathbb{E}\left(\varphi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right) = \mathbb{E}\left(\varphi\left(\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}\right)\right).$$

Proof. Let

$$d\mathbb{Q} = D\, d\mathbb{P}$$

be a probability measure which belongs to $\mathcal{E}^{\nu}$. Since the law of Y under $\mathbb{Q}$ is ν, we have

$$\mathbb{E}(D \mid Y) = \frac{d\nu}{d\mathbb{P}_Y}(Y).$$

Now, from Jensen's inequality

$$\varphi\left(\frac{d\nu}{d\mathbb{P}_Y}(Y)\right) \le \mathbb{E}(\varphi(D) \mid Y)$$

which implies

$$\mathbb{E}\left(\varphi\left(\frac{d\nu}{d\mathbb{P}_Y}(Y)\right)\right) \le \mathbb{E}(\varphi(D)). \square$$

Remark 11. Notice that since $\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}$ is assumed to be bounded, the value $\mathbb{E}\left(\varphi\left(\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}\right)\right)$ is finite.

Example 3. With $\varphi(x) = x^2$, we see that $\mathbb{P}^{\nu}$ is the minimal variance probability, i.e.

$$\inf_{\mathbb{Q}\in\mathcal{E}} \mathbb{E}\left(\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2\right) = \mathbb{E}\left(\left(\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}\right)^2\right).$$

With $\varphi(x) = x\ln x$, we see that it is also a minimal entropy measure.

Semimartingale Decomposition under the Minimal Probability

Our aim, now, is to develop stochastic calculus under the minimal probability $\mathbb{P}^{\nu}$. To do this, the first step is to compute the martingale density process of $\mathbb{P}^{\nu}$ with respect to $\mathbb{P}$. And then, we apply Girsanov's theorem. For this, we use the process $(\eta_t^y)_{0\le t<T, y\in\mathcal{P}}$ defined by Assumption 1 and the process $(\alpha_t^y)_{0\le t<T, y\in\mathcal{P}}$ defined in Lemma 1.

Lemma 4. *For $t < T$, $\mathbb{P}^{\nu}_{/\mathcal{F}_t}$ is absolutely continuous with respect to $\mathbb{P}_{/\mathcal{F}_t}$ and*

$$\mathbb{P}^{\nu}_{/\mathcal{F}_t} = \int_{\mathcal{P}} \eta_t^y \;\nu(dy)\; \mathbb{P}_{/\mathcal{F}_t}.$$

Proof. Since

$$\mathbb{P}^{\nu} = \frac{d\nu}{d\mathbb{P}_Y}(Y)\;\mathbb{P},$$

for $t < T$,

$$\mathbb{P}^{\nu}_{/\mathcal{F}_t} = \mathbb{E}\left(\frac{d\nu}{d\mathbb{P}_Y}(Y) \mid \mathcal{F}_t\right)\;\mathbb{P}_{/\mathcal{F}_t}.$$

Now, it is an immediate consequence of the definition of $(\eta_t^y)_{0\le t<T, y\in\mathcal{P}}$ that

$$\mathbb{E}\left(\frac{d\nu}{d\mathbb{P}_Y}(Y) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} \eta_t^y \;\nu(dy).\square$$

Theorem 11. *The process $(S_t)_{0\le t<T}$ is a $(\mathcal{F}, \mathbb{P}^{\nu})$ semimartingale and its decomposition is given by*

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \frac{\int_{\mathcal{P}} \eta_u^y \alpha_u^y \nu(dy)}{\int_{\mathcal{P}} \eta_u^y \nu(dy)} + M_t, \quad t < T \tag{13}$$

where $(M_t)_{0\le t<T}$ is a $(\mathcal{F}, \mathbb{P}^{\nu})$ local martingale such that

$$\langle M\rangle = \langle S\rangle.$$

Proof. The process

$$D_t = \int_{\mathcal{P}} \eta_t^y \ \nu\left(dy\right) \ , 0 \le t < T,$$

is the density process of $\mathbb{P}^\nu$ with respect to $\mathbb{P}$. By Lemma 1 and Fubini's theorem (we can apply it because of Assumption 3), we have for $0 \le t < T$, $1 \le i \le d$,

$$d\langle D, S^i\rangle_t = \left(\int_{\mathcal{P}} \eta_t^y \left(\sum_{i=1}^{d} \alpha_t^{y,j} d\langle S^i, S^j\rangle_t\right) \nu\left(dy\right)\right).$$

The result is then a consequence of Girsanov's theorem. □

Remark 12. The compensator of S under $\mathbb{P}^\nu$:

$$\int_0^t d\langle S\rangle_u \frac{\int_{\mathcal{P}} \eta_u^y \alpha_u^y \nu\left(dy\right)}{\int_{\mathcal{P}} \eta_u^y \nu\left(dy\right)}$$

represents the *information drift* given by the weak anticipation ν.

Connection with the Theory of Initial Enlargement of Filtration

In this paragraph, we show the link between the weak and the strong approach, precisely the following theorem completes the converse of Jacod's theorem (see Theorem 2).

Theorem 12. *Under the probability $\mathbb{P}^\nu$ the price process $(S_t)_{0\le t<T}$ is a semimartingale in the filtration $\mathcal{G}$, and its decomposition is given by*

$$S_t = s_0 + \int_0^t \langle S\rangle_u \alpha_u^Y + M_u, \qquad 0 \le t < T \tag{14}$$

where $(M_t)_{0\le t<T}$ is a $\mathbb{P}^\nu$ local martingale such that

$$\langle S\rangle = \langle M\rangle.$$

Remark 13. We recover the decomposition (14) from (13) with $\nu = \delta_Y$, nevertheless this is only formal because in our assumptions ν is not assumed to be random. This shows the analogy between Jacod's and Girsanov's theorem. This is very well explained in [46], where the author understood the Jacod's theorem as a Girsanov formula applied on a convenient product probability space .

Remark 14. The decomposition (13) can also be written

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \mathbb{E}^\nu\left(\alpha_u^Y \mid \mathcal{F}_u\right) + M_t$$

which is a posteriori explained from the decomposition (14) by the filtering theory.

Conditioned Stochastic Differential Equations

We define here the notion of conditioned stochastic differential equations. This notion has been introduced in [5]. The idea is to construct a stochastic differential equation whose distribution of the solution is the same as the law of the process $(S_t)_{0\leq t\leq T}$ under $\mathbb{P}^\nu$. This gives hence a tool to construct *minimal* models associated with weak anticipations.

Let us assume that the information drift $\left(\frac{\int_{\mathcal{P}} \alpha_t^y \eta_t^y \nu(dy)}{\int_{\mathcal{P}} \eta_t^y \nu(dy)}\right)_{0\leq t<T}$ is predictable with respect to the natural filtration of $(S_t)_{0\leq t\leq T}$, then there exists on the Wiener space of continuous functions $\mathbb{W}^d$ a predictable function F such that for all $t<T$,

$$\frac{\int_{\mathcal{P}} \alpha_t^y \eta_t^y \nu\left(dy\right)}{\int_{\mathcal{P}} \eta_t^y \nu\left(dy\right)} = F^\nu\left(t, \left(S_u\right)_{u\leq t}\right).$$

Let us assume furthermore that under the martingale measure $\mathbb{P}$, $(S_t)_{0\leq t\leq T}$ can be written

$$S_t = s_0 + \int_0^t \sigma\left(s, \left(S_u\right)_{u\leq s}\right) dW_s$$

where W is a standard Brownian motion and where σ is a predictable functional on the Wiener space valued in the space of $d\times d$ matrix.

Definition 8. *Let* $\left(\widetilde{\Omega}, \left(\mathcal{H}_t\right)_{0\leq t<T}, \mathbb{Q}\right)$ *be any filtered probability space on which a d-dimensional standard $\mathcal{H}-$adapted Brownian motion* $(\beta_t)_{0\leq t<T}$ *is defined. The stochastic differential equation*

$$X_t = s_0 + \int_0^t \left(\sigma^*\sigma\right)\left(s, \left(X_u\right)_{u\leq s}\right) F^\nu\left(s, \left(X_u\right)_{u\leq s}\right) ds + \int_0^t \sigma\left(s, \left(X_u\right)_{u\leq s}\right) d\beta_s \tag{15}$$

$t<T$*, is called the conditioned stochastic differential equation (in abbreviate CSDE) associated with the conditioning* (T,Y,ν).

Remark 15. By construction, the stochastic differential equation (15) has always a weak solution defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}^\nu)$.

Thanks to Yamada-Watanabe's theorem (see [43] pp. 368), we can now state an important transfer result:

Theorem 13. *Assume that the stochastic differential equation* (15) *enjoys the pathwise uniqueness property. Then* (15) *has a unique strong solution* $(X_t)_{0\leq t<T}$ *associated with the initial condition* $X_0=s_0$ *and the law of* $(X_t)_{0\leq t<T}$ *is the same as the law of* $(S_t)_{0\leq t\leq T}$ *under the minimal probability* $\mathbb{P}^\nu$ *associated with the conditioning* (Y,ν).

Connection with the Theory of Schrödinger Processes

Before we turn to examples of conditioning, in this paragraph we show how our results are closely related to some processes studied in quantum mechanics and called Schrödinger processes (see by e.g. [39]). We shall assume here that $(\Omega, \mathcal{F}, \mathbb{P})$ is the d-dimensional Wiener space of continuous paths, i.e. that Ω is the space of continuous functions $[0,T] \to \mathbb{R}^d$ that $\mathcal{F}$ is the natural filtration of the coordinate process and that $\mathbb{P}$ is the Wiener measure. Hence, for a continuous stochastic process Z (defined on any suitable probability space), $Y(Z)$ denotes the $\mathcal{P}$-valued random variable $\omega \to Y((Z_t(\omega))_{0 \le t \le T})$.

Assume now that we are interested in the following variational problem:

Problem: *Let ν be a probability on $\mathcal{P}$ which is equivalent to $\mathbb{P}_Y$ with a bounded density. On a general filtered probability space*

$$\left(\tilde{\Omega}, (\mathcal{H}_t)_{0 \le t \le T}, (W_t)_{0 \le t \le T}, \mathbb{Q}\right)$$

on which a standard Brownian motion W is defined, we search an adapted control Θ minimizing the action

$$\mathcal{A} = \frac{1}{2}\,\mathbb{E}^{\mathbb{Q}}\left(\int_0^T \Theta_s^2 ds\right)$$

under the constraints

$$\mathbb{Q}\left(Y\left(Z^{\Theta}\right) \in dy\right) = \nu\left(dy\right),$$

and

$$\mathbb{P}_{Z^{\Theta}} \sim \mathbb{P}$$

where $\mathbb{P}_{Z^{\Theta}}$ is the law of

$$Z_t^{\Theta} = \int_0^t \Theta_s ds + W_t,\ t \le T.$$

For a probability

$$\tilde{\mathbb{Q}} = \mathcal{E}\left(\int_0^{\cdot} \Lambda_u \cdot dX_u\right)_T \mathbb{P}$$

on the Wiener space, the relative entropy of $\tilde{\mathbb{Q}}$ with respect to $\mathbb{P}$ (assuming that the integral below is convergent) is given by

$$\frac{1}{2}\,\mathbb{E}^{\tilde{\mathbb{Q}}}\left(\int_0^T \Lambda_u^2\, du\right).$$

Now, since the function $x \mapsto x\ln(x)$ is strictly convex on the open set $(0,\infty)$, Proposition 5 can be applied, and we conclude:

Theorem 14. *Assume that the relative entropy of* ν *with respect to* $\mathbb{P}_Y$ *is finite, then the previous problem admits one and only one solution* Θ^* *and the law* $\mathbb{P}_{Z^{\Theta^*}}$ *of the corresponding process* Z^{Θ^*} *satisfies the following absolute continuity relationship*

$$\mathbb{P}_{Z^{\Theta^*}} = \xi\left(Y\right)\, \mathbb{P}$$

where $\xi = \frac{d\nu}{d\mathbb{P}_Y}$.

Remark 16. It is known from [22] that since

$$\frac{1}{2}\,\mathbb{E}^{\mathbb{Q}}\left(\int_0^T \left(\Theta_s^*\right)^2 ds\right) < +\infty$$

the following limit (called Nelson forward derivative) exists in L^2

$$\mathcal{D}_t Z^{\Theta^*} := \lim_{h \to 0+} \mathbb{E}^{\mathbb{Q}}\left(\frac{Z_{t+h}^{\Theta^*} - Z_t^{\Theta^*}}{h} \mid \mathcal{F}_t\right),\ t < T$$

and is equal to Θ^*. This is hence by analogy with classical mechanics that

$$L_t = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left(\left(\mathcal{D}_t Z^{\Theta^*}\right)^2\right)$$

is called a Lagrangian and

$$\mathcal{A} = \int_0^T L_t\, dt$$

an action integral.

In the case where $Y = X_T$, X being the coordinate process, then the previous theorem is well known: Z^{Θ^*} is Markov and is called a Schrödinger process (for further details on this case, we refer to Subsection 3.2).

3.2 Examples of Conditioning

We now give some examples of conditioning. These examples are the same as those studied in Subsection 2.2.

Stochastic Analysis

Here again (see Section 2.2.1.), we use stochastic analysis to obtain explicit computations for the conditioning of a functional of the coordinate process. We assume that

$$\mathcal{P} = \mathbb{R}^N$$

for some integer $N \geq 1$ and that Y belongs to $\left(\mathbb{D}^{1,2}\right)^N$.

Proposition 6. *Assume that $\xi := \frac{d\nu}{d\mathbb{P}_Y}$ admits a continuously differentiable version with bounded partial derivatives then, for $t < T$*

$$\frac{\int_{\mathbb{R}^d} \eta_t^y \alpha_t^y \nu(dy)}{\int_{\mathbb{R}^d} \eta_t^y \nu(dy)} = \mathbf{D}_t \ln \mathbb{E}(\xi(Y) \mid \mathcal{F}_t) = \mathbb{E}^\nu\left(\left(\frac{\nabla \xi}{\xi}\right)^* (Y) \mathbf{D}_t Y \mid \mathcal{F}_t\right).$$

Proof. Under these assumptions, we have

$$\mathbb{P}^\nu = \xi(Y)\ \mathbb{P}$$

and so for $t \leq T$,

$$\mathbb{P}^\nu_{/\mathcal{F}_t} = \mathbb{E}(\xi(Y) \mid \mathcal{F}_t)\ \mathbb{P}_{/\mathcal{F}_t}.$$

Now from the Clark-Ocone formula (see [12], [37] pp. 183 and [40] Proposition 1.3.5.)

$$\begin{aligned}\mathbb{E}(\xi(Y) \mid \mathcal{F}_t) &= 1 + \int_0^t \mathbb{E}(\mathbf{D}_s \xi(Y) \mid \mathcal{F}_s) \cdot dX_s \\ &= 1 + \int_0^t \mathbb{E}\left((\nabla \xi(Y))^*\ \mathbf{D}_s Y \mid \mathcal{F}_s\right) \cdot dX_s.\end{aligned}$$

Hence,

$$\frac{\int_{\mathbb{R}^N} \alpha_t^y \eta_t^y \nu(dy)}{\int_{\mathbb{R}^N} \eta_t^y \nu(dy)} = \frac{\mathbb{E}\left((\nabla \xi(Y))^* \mathbf{D}_t Y \mid \mathcal{F}_t\right)}{\mathbb{E}(\xi(Y) \mid \mathcal{F}_t)}$$

and the Bayes formula gives the expected result. □

Remark 17. Under the assumptions of the previous proposition, the formula for the compensator of S under $\mathbb{P}^\nu$, or the *information drift* takes hence the following nice form

$$\mathbf{D}_t \ln \int \frac{\nu(dy)}{\mathbb{P}_Y(dy)} \mathbb{P}(Y \in dy \mid \mathcal{F}_t).$$

Conditioning of the Terminal Value of a Diffusion

Now, we return (see subsection 2.2.2) to the case $\mathcal{P} = \mathbb{R}^d$ and $Y = S_T$ where the dynamics of $(S_t)_{0 \leq t \leq T}$ under $\mathbb{P}$ are given by

$$S_t = s_0 + \int_0^t \operatorname{diag}(S_u)\,\sigma(S_u)\,dB_u, \qquad 0 \leq t \leq T \tag{16}$$

where $(B_t)_{0 \leq t \leq T}$ is a d-dimensional standard Brownian motion whose filtration is $\mathcal{F}$, $s_0 \in \left(\mathbb{R}_+^*\right)^d$ and σ a positive definite symmetric bounded C^∞ with bounded partial derivatives function such that

$$\inf_{x \in \mathbb{R}^d} \|\sigma\sigma^*(x)\| \geq a > 0.$$

And we consider again the transition function p_t, $0 < t \leq T$. In this setting, we easily obtain by the same way as in Section 2.2.2.

Theorem 15. *(See [8]) Under the probability $\mathbb{P}^\nu$, the process $(S_t)_{0\leq t<T}$ admits the following semimartingale decomposition*

$$S_t = s_0 + \int_0^t (\tilde{\sigma}^* \tilde{\sigma})(S_u) \frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-u}(S_u,y)}{p_T(s_0,y)} \nu(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-u}(S_u,y)}{p_T(s_0,y)} \nu(dy)} du + \int_0^t \tilde{\sigma}(S_u)\, d\beta_u \tag{17}$$

where $(\beta_t)_{0\leq t\leq T}$ is a d-dimensional standard Brownian motion under $\mathbb{P}^\nu$ and

$$\tilde{\sigma}(x) = \operatorname{diag}(x)\, \sigma(x).$$

Remark 18. Let us now consider on a filtered probability space

$$\left(\tilde{\Omega}, (\mathcal{H}_t)_{0\leq t\leq T}, \mathbb{Q}\right)$$

which satisfies the usual conditions and on which is defined a d-dimensional standard Brownian motion $(W_t)_{0\leq t\leq T}$, the following stochastic differential equation

$$X_t = s_0 + \int_0^t (\tilde{\sigma}^* \tilde{\sigma})(X_u) \frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-u}(X_u,y)}{p_T(s_0,y)} \nu(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-u}(X_u,y)}{p_T(s_0,y)} \nu(dy)} du + \int_0^t \tilde{\sigma}(X_u)\, dW_u.$$

If this SDE enjoys the pathwise uniqueness property, then thanks to Theorem 13, it admits a strong solution whose law is the same as the law of S under $\mathbb{P}^\nu$. That is why we can speak of *minimal* model for the price process including the weak information ν on the price at the date T.

More generally, in [8], we have studied the conditioning of the value at time T of any Markov process. In this case, under the minimal probability measure, the price process is the so called Doob's transform of the starting process.

Precisely, here are the results. Let us denote by $\mathcal{L}$ the (extended) generator of S under $\mathbb{P}$. We assume that the domain $D(\mathcal{L})$ contains C^2 functions with compact support and that for such functions ϕ,

$$\mathcal{L}\phi(x) = \frac{1}{2}\sum_{i,j=1}^d a_{i,j}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d\setminus\{0\}} (\phi(x+y) - \phi(x) - y\cdot\nabla\phi(x))\, N(x,dy) \tag{18}$$

where $a(x)$ is a smooth function with values in nonnegative definite symmetric $d\times d$ matrices, and $N(x,dy)$ is the Lévy kernel of S (see [17]).

In this setting, there is a function d such that $\mathbb{E}[\xi(S_T)|\mathcal{F}_t] = d(t,S_t)$. Note that we assumed that ξ is a.s. bounded and strictly positive, so that $d(t,x)$ is strictly positive for almost all t and x.

Theorem 16. *(See [8]) $\mathbb{P}^\nu$ solves the martingale problem associated to $\mathcal{L}^\nu$ and the initial distribution δ_{s_0}, where*

$$\begin{aligned}\mathcal{L}^{\nu}\phi(t,x) &= \frac{1}{d(t,x)}\bar{\mathcal{L}}(\phi d)(t,x) \\ &= \frac{1}{d(t,x)}\frac{\partial(\phi d)}{\partial t}(t,x) + \frac{1}{d(t,x)}\mathcal{L}(\phi(t,.)d(t,.))(x)\end{aligned}$$

for any ϕ such that $\phi \in D(\bar{\mathcal{L}})$ and $\phi d \in D(\bar{\mathcal{L}})$, where $\bar{\mathcal{L}}$ is the generator of the space-time process (t, S_t) which is given by:

$$\bar{\mathcal{L}}\phi(t,x) = \frac{\partial \phi}{\partial t}(t,x) + \mathcal{L}(\phi(t,.))(x).$$

Conditioning of the Perpetuity

In this paragraph, we return to the case $\mathcal{P}$ $=\mathbb{R}$ and assume that the dynamics of S under $\mathbb{P}$ is one-dimensional and given by

$$S_t = \exp\left(B_t - \frac{1}{2}t\right),\ t \geq 0$$

where $(B_t)_{0\leq t\leq T}$ is a one-dimensional standard Brownian motion whose filtration is $\mathcal{F}$. We have the following conditioning formula for the functional $\int_0^{+\infty} S_u^2 du$.

Theorem 17. *(See [5], [7]) Assume that ν admits with respect to the function $\frac{e^{-\frac{1}{2y}}}{y^{3/2}}1_{y>0}$ a C^2 density $\xi : \mathbb{R}_+^* \to \mathbb{R}_+$ which is almost surely bounded. The decomposition of S under $\mathbb{P}^\nu$ is then given by*

$$S_t = 1 + \int_0^t S_v \left(1 - 2\frac{\int_0^{+\infty} e^{-u}u^{1/2}\xi\left(\int_0^v S_s^2 ds + \frac{S_v^2}{2u}\right)du}{\int_0^{+\infty} e^{-u}u^{-1/2}\xi\left(\int_0^v S_s^2 ds + \frac{S_v^2}{2u}\right)du}\right) dv + \beta_t\ ,\ t\geq 0 \tag{19}$$

where $(\beta_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}^\nu$.

Conditioning of Hitting Times

Let $(B_t)_{0\leq t\leq T}$ be a one-dimensional standard Brownian motion whose filtration is $\mathcal{F}$. In this paragraph, we give the conditioning formula associated with

$$T_a = \inf\ \{t \geq 0\ ,B_t = a\}\ , a > 0.$$

Here, we assume that ν is a Borel measure defined on $\mathbb{R}_+^*$ by

$$\nu\,(dt) = \left(\int_0^{+\infty} e^{-t\frac{\delta^2}{2}+\delta a}\ m\,(d\delta)\ \right)\gamma\,(dt) \tag{20}$$

with m a probability measure on $\mathbb{R}_+^*$ such that $\int_0^{+\infty} \delta^2\ m\,(d\delta) < +\infty$ and

$$\gamma\,(dt) = \frac{a}{\sqrt{2\pi t^3}}\ e^{-\frac{a^2}{2t}}dt,\ t > 0.$$

We recall that γ is the law of T_a under $\mathbb{P}$ (see [43] pp. 107).

Remark 19. We take ν under the form (20) in order to use directly exponential martingales (see the expression (21)). Otherwise, we would have to invert the formula (20), which appears to be rather complicated.

We have then the following conditioning formula.

Theorem 18. *(See [5]) Under the probability $\mathbb{P}^\nu$ defined on $\mathcal{F}_{T_a}$ by*

$$\mathbb{P}^\nu_{/\mathcal{F}_{T_a}} = \int_0^{+\infty} e^{-T_a \frac{\delta^2}{2} + \delta a} \, m\left(d\delta\right) \, \mathbb{P}_{/\mathcal{F}_{T_a}}, \tag{21}$$

the process $(B_t)_{0 \le t \le T_a}$ is a semimartingale whose decomposition is given by

$$B_t = \int_0^t \frac{\int_0^{+\infty} \delta \, e^{-s\frac{\delta^2}{2} + \delta B_s} \, m\left(d\delta\right)}{\int_0^{+\infty} e^{-s\frac{\delta^2}{2} + \delta X_s} \, m\left(d\delta\right)} ds + \beta_t, \quad t \le T_a \tag{22}$$

where β is a standard Brownian motion under $\mathbb{P}^\nu$.

Example 4. Let $\alpha > 0$. With $m = \frac{1}{2}\delta_\alpha + \frac{1}{2}\delta_{-\alpha}$, decomposition (22) becomes

$$B_t = \alpha \int_0^t \tanh\left[\alpha B_s\right] ds + \beta_t.$$

3.3 Pathwise Conditioning

Before we conclude this section with some general comments on the conditioning, we would like to present briefly another kind of conditioning. Precisely, let us assume that an insider is in the following position: He knows that with probability one

$$\forall t > 0, \quad S_t \in \mathcal{O}$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a non-empty, open, simply connected, and relatively compact set (the boundary of $\mathcal{O}$ shall be denoted by $\partial\mathcal{O}$).

Our question is now: Can we give to this insider a *minimal* model which takes into account this information ?

We shall answer this question in the case where, under the martingale measure $\mathbb{P}$, S is an homogeneous diffusion with elliptic generator $\mathcal{L}$ (the general case where S is only a local martingale seems to need more works).

For this, we proceed in two steps:

1. First, we give a sense to the following probability measure

$$\mathbb{P}^* = \mathbb{P}(\cdot \mid \forall t > 0, S_t \in \mathcal{O})$$

2. Secondly, we compute the semimartingale decomposition of S under $\mathbb{P}^*$.

To fulfill this program, we first define $\mathbb{P}^*$, as being the weak limit when $t \to +\infty$ (it is shown just below that it exists under suitable assumptions) of the following sequence of probabilities

$$\mathbb{P}^t = \mathbb{P}(\cdot \mid T_{\partial\mathcal{O}} = t), \quad t > 0$$

where

$$T_{\partial\mathcal{O}} = \inf\{t \leq 0, \ S_t \notin \mathcal{O}\}.$$

It is easily seen, by the strong Markov property of S under $\mathbb{P}$ that we have:

$$\mathbb{P}^t_{/\mathcal{F}_u} = \frac{g(S_u, t-u)}{g(s_0, t)} \, \mathbb{P}_{/\mathcal{F}_u}, \quad u < T_{\partial\mathcal{O}}$$

where g is defined by

$$\mathbb{P}(T_{\partial\mathcal{O}} \in dt \mid S_0 = x) = g(x,t)dt, \quad (x,t) \in \mathcal{O} \times \mathbb{R}^+.$$

Of course, implicitly, we assume that g such characterized is well defined and positive. Moreover, we shall assume that it is smooth.

Proposition 7. *We have, for all bounded and $\mathcal{F}_\infty$-measurable random variable F,*

$$\lim_{t \to +\infty} \mathbb{E}\left(\frac{g(S_u, t-u)}{g(s_0, t)} \, F\right) = \mathbb{E}\left(e^{\lambda_1 u} \psi_1(S_u) \, F\right)$$

where λ_1 is the smallest (positive) eigenvalue and ψ_1 the corresponding eigenfunction of the Dirichlet problem:

$$\frac{1}{2}\mathcal{L}\psi + \lambda\psi = 0 \text{ on } \mathcal{O},$$

and

$$\psi_{/\partial\mathcal{O}} = 0, \quad \psi_1(s_0) = 1.$$

Proof. In fact, this is a direct consequence of the theory of Dirichlet problems for elliptic operators on relatively compact open sets (these problems are widely discussed in [42]). Indeed, from this theory, g which solves the following problem:

$$\frac{\partial g}{\partial t} = \frac{1}{2}\mathcal{L}g \text{ on } \mathcal{O} \times \mathbb{R}^+ \text{ and } g(x,t) = \delta_0 \text{ on } \partial\mathcal{O} \times \mathbb{R}^+,$$

can be expanded on $\mathcal{O} \times \mathbb{R}^+$:

$$g(x,t) = \sum_{i=1}^{+\infty} e^{-\lambda_i t} \tilde{\psi}_i(x)$$

where $0 < \lambda_1 < ... < \lambda_n < ...$ are the eigenvalues and $\psi_1, ..., \psi_n, ...$ corresponding eigenfunctions of the Dirichlet problem

$$\frac{1}{2}\mathcal{L}\psi + \lambda\psi = 0 \text{ on } \mathcal{O}, \text{ and } \psi_{/\partial\mathcal{O}} = 0.$$

The proof follows then almost immediately after straightforward considerations. □

From this, we deduce first that the following absolute continuity relationship holds:

$$\mathbb{P}^*_{/\mathcal{F}_u} = e^{\lambda_1 u} \psi_1(S_u)\, \mathbb{P}_{/\mathcal{F}_u}, \quad u < T_{\partial \mathcal{O}}$$

and secondly from Girsanov's theorem the semimartingale decomposition of S under $\mathbb{P}^*$ (up to $T_{\partial\mathcal{O}}$, but notice that actually $T_{\partial\mathcal{O}} = +\infty$, $\mathbb{P}^*$ a.s.).

Example 5. As a consequence of this, we can deduce for instance that a one-dimensional standard Brownian motion B conditioned by the event $\{\forall t \geq 0, B_t \in [a, b]\}$ with $a < 0 < b$ is a Jacobi diffusion.

Example 6. A one-dimensional Brownian motion started from $a > 0$ and conditioned by the event $\{\forall t \geq 0, B_t > 0\}$ is a 3-dimensional Bessel process (which is clearly related to Pitman's theorem).

3.4 Comments

1. In this section, we did not try to perform the most general setting in which our conditioning technique works. We preferred focusing on the ideas of the construction of $\mathbb{P}^\nu$. In fact Assumption 1 is even not necessary. Indeed, our conditioning technique stems of the semimartingale decomposition of the price process under the minimal probability $\mathbb{P}^\nu$. Now, since

$$\mathbb{P}^\nu = \frac{d\nu}{d\mathbb{P}_Y}(Y)\ \mathbb{P},$$

   from Girsanov's theorem it is clear that the price process remains a semimartingale under $\mathbb{P}^\nu$ even if Assumption 1 is not satisfied. But this decomposition is not as explicit as the decomposition (13). We have worked in this section under this assumption in order to provide closed formulas and in order to show the link with the theory of initial enlargement of filtration.
2. We can extend our work to the case where the measure ν is singular with respect to $\mathbb{P}_Y$. But for this, we need further assumptions. Indeed, the family of conditional probabilities $\mathbb{P}(\cdot \mid Y = y)$ is only defined $\mathbb{P}_Y$-a.s., so that the formula

$$\mathbb{P}^\nu = \int_{\mathcal{P}} \mathbb{P}\left(\cdot \mid Y = y\right) \nu\left(dy\right) \tag{23}$$

   does not make sense when ν is singular with respect to $\mathbb{P}_Y$. Nevertheless, for instance, let us assume that we can choose a *canonical* version of the regular conditional distributions given Y such that the map $y \to \mathbb{P}(\cdot \mid Y = y)$ is continuous in the weak topology on probability measures. In this case, we can still define $\mathbb{P}^\nu$ by the formula 23, and it is not difficult to see that the law of Y under $\mathbb{P}^\nu$ is given by ν.

4 Utility Maximization with Weak Information

We now turn to financial applications of the conditioning. Precisely, we try to give a quantitative financial value to the weak information (Y, ν) . This value should satisfy the following rule: Less the information is precise, less this value is. For instance the *minimal* information is $\nu = \mathbb{P}_Y$ because under $\mathbb{P}$ the price process $(S_t)_{0 \le t < T}$ is a local martingale and the *maximal* information is obtained at the limit with $\nu = \delta_y$ for $y \in \mathcal{P}$. Since the probability ν is assumed to be equivalent to $\mathbb{P}_Y$, we shall see that it implies that there is no arbitrage (in the sense defined Section 1); this is one of the main difference with the strong information setting. Throughout this section we shall denote by ξ a version of the density of ν with respect to $\mathbb{P}_Y$. Furthermore, we shall again assume here that ξ can be chosen bounded.

4.1 Portfolio Optimization Problem

We first introduce the set of insiders which are weakly informed on the functional Y and define what will be called the financial value of the weak information (Y, ν). Precisely, let $\mathcal{E}^\nu$ be the set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ such that:

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$
2. $\mathbb{Q}\,(Y \in dy) = \nu\,(dy)$.

The financial market model associated with an element $\mathbb{Q}$ of $\mathcal{E}^\nu$ is

$$\left(\Omega, (\mathcal{F}_t)_{0 \le t < T}, \mathbb{Q}, (S_t)_{0 \le t \le T}\right). \tag{24}$$

It is then clear that there is no arbitrage on this market because $\mathbb{Q} \sim \mathbb{P}$ and S is a local martingale under $\mathbb{P}$. Now, the portfolio optimization problem associated with (24) is the following:

Portfolio optimization problem: *The insider's portfolio optimization problem is to find*

$$\sup_{\Theta \in \mathcal{A}_{\mathcal{F}}(S)} \mathbb{E}^{\mathbb{Q}} \left(U \left(x + \int_0^t \Theta_u dS_u \right) \right)$$

$x > 0$ *being the initial endowment of the insider.*

We recall here that U is a utility function and that $\mathcal{A}_{\mathcal{F}}\,(S)$ is the set of **adapted** admissible strategies (see Section 1).

Definition 9. *We define the financial value of the weak information* (Y, ν) *as being*

$$u\,(x, \nu) := \inf_{\mathbb{Q} \in \mathcal{E}^\nu} \sup_{\Theta \in \mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}} \left(U \left(x + \int_0^T \Theta_u dS_u \right) \right).$$

In other words, we define the financial value of the weak in formation as being the lowest increase in utility that can be gained by the insider from this extra knowledge .

Value of the Weak Information in a Complete Market

We assume in this subsection that the market is complete. The following proposition, which is just, by convex duality, a consequence of proposition 5 shows the universal property of $\mathbb{P}^{\nu}$ ($\mathbb{P}^{\nu}$ does not depend on the utility function used by the insider) among the other elements of $\mathcal{E}^{\nu}$. It also gives the exact value of the weak information. We use here classical results on martingale dual approach in a complete market (see [31] and [32]).

Theorem 19. *Assume that integrals below are convergent. Then for each initial investment $x > 0$,*

$$\begin{aligned} u(x,\nu) &= \sup_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\nu}\left(U\left(x+\int_0^T \Theta_u dS_u\right)\right) \\ &= \int_{\mathcal{P}} (U\circ I)\left(\frac{\Lambda(x)}{\xi(y)}\right)\nu(dy) \end{aligned}$$

where $\Lambda(x)$ is defined by

$$\int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\mathbb{P}_Y(dy) = x.$$

Moreover, under $\mathbb{P}^{\nu}$ the optimal wealth process is given by

$$V_t = \int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\eta_t^y \mathbb{P}_Y(dy),$$

and the corresponding number of parts invested in the risky asset S by

$$\Theta_t = \int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\eta_t^y \alpha_t^y \mathbb{P}_Y(dy).$$

Proof. We will use a duality argument, as is now classical in this type of problem. Let $\mathbb{Q} = D\mathbb{P} \in \mathcal{E}^{\nu}$. We first proceed to rewrite

$$\sup_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}}\left(U\left(x+\int_0^T \Theta_u\cdot dS_u\right)\right)$$

differently. In fact, from classical results on complete markets, the Lagrangian associated with the optimization problem above is known to be given by

$$L(y) = xy + \mathbb{E}^{\mathbb{P}}\left(D\tilde{U}\left(\frac{y}{D}\right)\right) \qquad (y>0)$$

where we recall that $\tilde{U}$ is the convex conjugate of U. Moreover

$$\sup_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}}\left(U\left(x+\int_0^T \Theta_u\cdot dS_u\right)\right) = \inf_{y>0} L(y).$$

Hence we have

$$u(x,\nu) = \inf_{y>0} \left\{ xy + \inf_D \mathbb{E}^{\mathbb{P}} \left(D\tilde{U}\left(\frac{y}{D}\right)\right)\right\}$$

where D runs through the densities $d\mathbb{Q}/d\mathbb{P}$, $\mathbb{Q} \in \mathcal{E}^{\nu}$. Now the function $z \mapsto z\tilde{U}\left(\frac{y}{z}\right)$ is convex for fixed y, and by proposition 5, we obtain

$$u(x,\nu) = \inf_{y>0} \left\{ xy + \mathbb{E}^{\mathbb{P}} \left(\xi(Y)\tilde{U}\left(\frac{y}{\xi(Y)}\right)\right)\right\}.$$

The function $y \mapsto xy + \mathbb{E}^{\mathbb{P}} \left(\xi(Y)\tilde{U}\left(\frac{y}{\xi(Y)}\right)\right)$ inherits from $\tilde{U}$ the properties to be strictly convex, continuously differentiable and to tend to $+\infty$ as $y \to +\infty$; hence there exists $\Lambda(x)$ that realizes the inf, and $\Lambda(x)$ is given by

$$x - \mathbb{E}^{\mathbb{P}} \left(\xi(Y) I \left(\frac{\Lambda(x)}{\xi(Y)}\right)\right) = 0.$$

From this proof, it also follows immediately that if we denote by V the optimal wealth process under $\mathbb{P}^{\nu}$ then we have:

$$V_T = I\left(\Lambda\left(x\right)\frac{d\mathbb{P}_Y}{d\nu}\left(Y\right)\right)$$

which implies the second part of the theorem. □

As in the case of a strong information on Y, we give explicitly the formulas for the most commonly used utility functions.

Example 7. (See [5], [8]).

1. Let $\alpha \in (0,1)$ and $U(x) = \frac{x^{\alpha}}{\alpha}$ then

$$u\left(x,\nu\right) = \frac{x^{\alpha}}{\alpha}\left[\int_{\mathcal{P}}\left(\frac{d\nu}{d\mathbb{P}_Y}\left(y\right)\right)^{\frac{1}{1-\alpha}}\mathbb{P}\left(Y\in dy\right)\right]^{1-\alpha}$$

2. Let $U(x) = \ln x$ then

$$u\left(x,\nu\right) = \ln x + \int_{\mathcal{P}}\frac{d\nu}{d\mathbb{P}_Y}\left(y\right)\ln\frac{d\nu}{d\mathbb{P}_Y}\left(y\right)\mathbb{P}\left(Y\in dy\right).$$

Remark 20. $u(x,\nu)$ represents the minimal value of the terminal utility for an insider which is weakly informed on the functional Y. It will be shown later that we always have

$$u\left(x,\nu\right) \geq U\left(x\right)$$

and that the equality takes place for $\nu = \mathbb{P}_Y$. Hence, it is perhaps more natural (but of course equivalent) to define the value of the additional information as being

$$v\left(x,\nu\right) := u\left(x,\nu\right) - U\left(x\right).$$

From an intuitive point of view, the more the anticipation is precise, the greater $v(x,\nu)$ will be. For instance, in the case of a logarithmic utility, $v(x,\nu)$ is the relative entropy of ν with respect to $\mathbb{P}_Y$.

Remark 21. We have the following interesting additivity property: If an insider is weakly informed on two variables Y_1 and Y_2 which are independent under $\mathbb{P}$ then

$$\mathcal{E}^{\nu_1 \otimes \nu_2} = \mathcal{E}^{\nu_1} \cap \mathcal{E}^{\nu_2}$$

and, again in the case of a logarithmic function

$$v\left(x, \nu_1 \otimes \nu_2\right) = v\left(x, \nu_1\right) + v\left(x, \nu_2\right).$$

Value of the Weak Information in an Incomplete Market

We now turn to the case where the market is incomplete, i.e. we assume that S does not enjoy the PRP. Moreover, we make the following additional assumption on the asymptotic elasticity of our utility function (see [34]):

$$\limsup_{x \to \infty} \frac{xU'(x)}{U(x)} < 1.$$

This last assumption allows to use the classical duality methods (see [32] and [34]).

Theorem 20. *(See [8]) For each initial investment $x > 0$,*

$$u\left(x, \nu\right) = \inf_{y>0} \left(\left(\inf_{\pi \in \mathcal{D}} \int_{\mathcal{P}} \widetilde{U}\left(y\pi\left(u\right)\right) \nu\left(du\right) \right) + xy \right)$$

where

$$\mathcal{D} = \{ \frac{d\widetilde{\mathbb{P}}_Y}{d\nu}, \ \widetilde{\mathbb{P}} \in \mathcal{M}\left(S\right) \}.$$

Proof. First fix $\mathbb{Q} \in \mathcal{E}^{\nu}$, and let

$$u_{\mathbb{Q}}(x, \nu) = \sup_{\Theta \in \mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}} \left[U \left(x + \int_0^T \Theta_u \cdot dS_u \right) \right].$$

According to Theorem 2.2 in [34], we have

$$u_{\mathbb{Q}}(x, \nu) = \inf_{y>0} \left\{ xy + \inf_{\widetilde{\mathbb{P}} \in \mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}} \left[\widetilde{U} \left(y \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right] \right\}.$$

We deduce that our function u is given by

$$u(x, \nu) = \inf_{\widetilde{\mathbb{P}} \in \mathcal{M}(S)} \inf_{y>0} \left\{ xy + \inf_{\mathbb{Q} \in \mathcal{E}^{\nu}} \mathbb{E}^{\mathbb{Q}} \left[\widetilde{U} \left(y \frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right] \right\}.$$

Now, the optimal $\mathbb{Q}$ associated with a fixed $\widetilde{\mathbb{P}}$ is given by $\widetilde{\xi}(Y)\widetilde{\mathbb{P}}$, where $\widetilde{\xi} = \frac{d\nu}{d\widetilde{\mathbb{P}}_Y}$. Hence,

$$u(x,\nu) = \inf_{y>0} \inf_{\widetilde{\mathbb{P}}\in\mathcal{M}(S)} \left\{ xy + \widetilde{\mathbb{E}}\left[\widetilde{\xi}(Y)\widetilde{U}\left(\frac{y}{\widetilde{\xi}(Y)}\right)\right]\right\}$$
$$= \inf_{y>0} \inf_{\pi\in\mathcal{D}} \left\{ xy + \widetilde{\mathbb{E}}\left[\frac{1}{\pi(Y)}\tilde{U}(y\pi(Y))\right]\right\}$$

and we conclude with straightforward computations. □

Remark 22. We always have

$$u(x,\nu) \geq U(x).$$

Indeed, since $\widetilde{U}$ is a convex function, for $\xi \in \mathcal{D}$ and $y > 0$,

$$\int_{\mathcal{P}} \widetilde{U}(y\xi(u))\,\nu(du) \geq \widetilde{U}(y)$$

and hence,

$$u(x,\nu) \geq \inf_{y>0}\left(\widetilde{U}(y) + xy\right) = U(x).$$

Moreover, if $1 \in \mathcal{D}$, i.e. there exists $\widetilde{\mathbb{P}} \in \mathcal{M}(S)$ such that

$$\widetilde{\mathbb{P}}(Y \in dy) = \nu(dy)$$

then

$$u(x,\nu) = U(x).$$

It would be really interesting to know more about the quantity

$$\inf_{\pi\in\mathcal{D}} \int_{\mathcal{P}} \widetilde{U}(y\pi(u))\,\nu(du)$$

which seems to be hard to evaluate, even in the simplest examples of incomplete markets (Stochastic volatility models). Nevertheless, in the case $Y = S_T$, we can deduce from the previous proposition the following inequality.

Proposition 8. *Assume that $\mathcal{P} = \mathbb{R}^d$ and $Y = S_T$. Then we have for $x > 0$*

$$u(x,\nu) \geq \int_{\left(\mathbb{R}_+^*\right)^d} U(\alpha + \beta\cdot u)\,\nu(du)$$

where $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+^d$ are defined by

$$\frac{x\int_{\left(\mathbb{R}_+^*\right)^d} U'(\alpha+\beta\cdot u)\,\nu(du)}{\int_{\left(\mathbb{R}_+^*\right)^d} (\alpha+\beta\cdot u)\,U'(\alpha+\beta\cdot u)\,\nu(du)} = 1$$

$$\frac{x\int_{\left(\mathbb{R}_+^*\right)^d} u\,U'(\alpha+\beta\cdot u)\,\nu(du)}{\int_{\left(\mathbb{R}_+^*\right)^d} (\alpha+\beta\cdot u)\,U'(\alpha+\beta\cdot u)\,\nu(du)} = S_0.$$

Proof. Let $y > 0$. Let us consider the convex functional

$$\begin{array}{l} L : \mathcal{V} \to \mathbb{R}_+ \cup \{+\infty\} \\ \quad \xi \ \to \int_{\mathbb{R}_+^{*\,d}} \widetilde{U}\left(y\xi\left(u\right)\right)\nu\left(du\right) \end{array}$$

defined on the convex set

$$\mathcal{V} = \{\xi \geq 0,\ \int_{\mathbb{R}_+^{*\,d}} \xi\left(u\right)\nu\left(du\right) = 1,\ \int_{\mathbb{R}_+^{*\,d}} u\xi\left(u\right)\nu\left(du\right) = S_0\}.$$

Since $\mathcal{D} \subset \mathcal{V}$, we have

$$\inf_{\xi\in\mathcal{D}} \int_{\mathbb{R}_+^{*\,d}} \widetilde{U}\left(y\xi\left(u\right)\right)\nu\left(du\right) \geq \inf_{\xi\in\mathcal{V}} \int_{\mathbb{R}_+^{*\,d}} \widetilde{U}\left(y\xi\left(u\right)\right)\nu\left(du\right).$$

Now, because L is convex, in order to find the minimum of L on $\mathcal{V}$, it suffices to find a critical point. An easy computation shows that such a critical point $\widetilde{\xi}$ must satisfy

$$\int_{\mathbb{R}_+^{*\,d}} \widetilde{U}'\left(y\widetilde{\xi}\left(u\right)\right)\eta\left(u\right)\nu\left(du\right) = 0$$

for all η such that

$$\int_{\mathbb{R}_+^{*\,d}} \eta\left(u\right)\nu\left(du\right) = 0,\ \int_{\mathbb{R}_+^{*\,d}} u\eta\left(u\right)\nu\left(du\right) = 0.$$

This implies

$$\widetilde{\xi}\left(u\right) = \frac{1}{y}U'\left(\alpha\left(y\right) + \beta\left(y\right)\cdot u\right)$$

where $\alpha\left(y\right) \in \mathbb{R}_+$ and $\beta\left(y\right) \in \mathbb{R}_+^d$ are defined by

$$\int_{\mathbb{R}_+^{*\,d}} U'\left(\alpha\left(y\right) + \beta\left(y\right)\cdot u\right)\nu\left(du\right) = y$$

$$\int_{\mathbb{R}_+^{*\,d}} u\,U'\left(\alpha\left(y\right) + \beta\left(y\right)\cdot u\right)\nu\left(du\right) = y\,S_0$$

(see Remark 23 below). Hence

$$u\left(x,\nu\right) \geq \inf_{y>0}\left(\int_{\mathbb{R}_+^d} \widetilde{U}\left(U'\left(\alpha\left(y\right) + \beta\left(y\right)\cdot u\right)\right)\nu\left(du\right) + xy\right),$$

but for all $z > 0$

$$\widetilde{U}\left(U'\left(z\right)\right) = U\left(z\right) - zU'\left(z\right),$$

hence

$$u\left(x,\nu\right) \geq \inf_{y>0}\left(\int_{\mathbb{R}_+^d} U\left(\alpha\left(y\right) + \beta\left(y\right)\cdot u\right)\nu\left(du\right) - \left(\alpha\left(y\right) + \beta\left(y\right)\cdot u\right)y + xy\right).$$

The previous infimum is clearly attained for $y > 0$ such that

$$\alpha(y) + \beta(y) \cdot S_0 = x$$

which gives the expected result. □

Remark 23. $\alpha(y)$ and $\beta(y)$ are well defined by the formulas above. Indeed, according to a classical theorem of Hadamard, the application

$$(\alpha, \beta) \mapsto \left(\int_{(\mathbb{R}_+^*)^d} U'(\alpha + \beta \cdot u)\,\nu(du), \int_{(\mathbb{R}_+^*)^d} u\, U'(\alpha + \beta \cdot u)\,\nu(du) \right)$$

induces a diffeomorphism from $(\mathbb{R}_+^* \times (\mathbb{R}_+^*)^d) \setminus (0,0)$ onto itself because it is proper with an everywhere non-singular differential. Moreover, since the function $y \mapsto \alpha(y) + \beta(y) \cdot S_0$ maps intervals into intervals, there exists y such that $\alpha(y) + \beta(y) \cdot S_0 = x$.

4.2 Study of a Minimal Markov Market

In this section, we make a complete study of the market

$$\left(\Omega, (\mathcal{F}_t)_{t \leq T}, (S_t)_{t \leq T}, \mathbb{P}^\nu \right)$$

in a Markov setting. Precisely, we consider the case where S is one-dimensional and $Y = S_T$ and we assume furthermore that there exist a bounded C^∞ function with bounded derivatives $\sigma : \mathbb{R}_+ \to [a, +\infty[$ $(a > 0)$ and a $\mathbb{P}$-standard Brownian motion $(B_t)_{0 \leq t \leq T}$ whose filtration is $\mathcal{F}$ such that

$$dS_t = S_t \sigma(S_t)\, dB_t\,, 0 \leq t \leq T. \tag{25}$$

From Theorem 15, we get:

Proposition 9. *Under the minimal probability* $\mathbb{P}^\nu$,

$$S_t = s_0 + \int_0^t S_u^2 \sigma(S_u)^2 \frac{\partial}{\partial x} \ln \varphi(u, S_u)\; du + \int_0^t S_u \sigma(S_u)\, d\beta_u\,, 0 \leq t \leq T \tag{26}$$

where β *is a* $(\mathcal{F}, \mathbb{P}^\nu)$ *standard Brownian motion and* φ *the solution of the partial differential equation*

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} x^2 \sigma^2(x) \frac{\partial^2 \varphi}{\partial x^2} = 0 \tag{27}$$

associated with the limit condition

$$\varphi(T, x) = \xi(x).$$

After the study of the dynamics of the price process under $\mathbb{P}^{\nu}$, we now turn to optimal strategies. As it is shown in the next proposition, the optimal wealth process and the corresponding strategy associated with the model (26) are Markovian. We stress the fact that this is not the case in all generality, and that this property is once again characteristic of the minimal probability $\mathbb{P}^{\nu}$. In fact, the following proposition is a direct consequence of Theorem 19.

Proposition 10. *In the market*

$$\left(\Omega, (\mathcal{F}_t)_{t\leq T}, (S_t)_{t\leq T}, \mathbb{P}^{\nu}\right)$$

the optimal wealth process $(V_t)_{0\leq t\leq T}$ *is Markovian and can be written*

$$V_t = h(t, S_t)$$

where

$$h(t,y) = \mathbb{E}\left(I\left(\frac{\Lambda(x)}{\xi(S_T)}\right) \mid S_t = y\right).$$

Remark 24. It is interesting to note that we can deduce from this that the optimal proportion process is given by

$$\Pi_t(x) = S_t\, \pi(t, S_t)$$

where π solves the partial differential equation

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(x^2\sigma^2(x)\frac{\partial \pi}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(x^2\sigma^2(x)\pi^2\right) = 0.$$

This equation, called the Burgers equation is well-known in fluid mechanics and particularly in aerodynamics (see [13]).

Let us now assume moreover that the volatility σ is a strictly positive constant. Hence

$$dS_t = \sigma S_t dB_t$$

which implies

$$S_t = S_0 e^{\sigma B_t - \frac{\sigma^2}{2}t}$$

hence, by a change of variable, a weak information on S_T is equivalent to a weak information on the functional B_T, precisely

Proposition 11. *Under the minimal probability* $\mathbb{P}^{\nu}$*,*

$$dS_t = \sigma S_t dB_t\,,\ t < T \tag{28}$$

where B *satisfies*

$$dB_t = \frac{\int_{-\infty}^{+\infty}\left(\frac{y-B_t}{T-t}\right) e^{\frac{y^2}{2T} - \frac{(y-B_t)^2}{2(T-t)}}\nu(dy)}{\int_{-\infty}^{+\infty} e^{\frac{y^2}{2T} - \frac{(y-B_t)^2}{2(T-t)}}\nu(dy)} dt + d\beta_t,\ t < T \tag{29}$$

β *being a* $(\mathcal{F}, \mathbb{P}^{\nu})$ *standard Brownian motion.*

For a complete study of the stochastic differential equation (29) (which enjoys the pathwise uniqueness property), we refer to [5]. An immediate corollary of Proposition 10 is:

Proposition 12. *In the market*

$$\left(\Omega, (\mathcal{F}_t)_{t\leq T}, (S_t)_{t\leq T}, \mathbb{P}^{\nu}\right)$$

the optimal wealth process is given by

$$V_t = h\,(t, B_t)$$

where B is defined by (29) and h given by

$$h(t,y) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} I\left(\frac{\Lambda(x)}{\xi(z)}\right) e^{-\frac{(z-y)^2}{2(T-t)}}\, dz.$$

We conclude this section with a particular case of the above market, precisely we study the case where ν is a Gaussian:

$$\nu\,(dx) = \frac{e^{-\frac{(x-m)^2}{2s^2}}}{\sqrt{2\pi}s} dx$$

with $m \in \mathbb{R}$ and $s^2 \leq T$. In this case, B is a Gaussian process and then S a log-normal process. Precisely, straightforward computations lead to

$$dS_t = \sigma S_t dB_t$$

where B satisfies

$$dB_t = \frac{\left(s^2 - T\right) B_t + mT}{\left(s^2 - T\right) t + T^2} dt + d\beta_t.$$

Remark 25. For $s^2 = T$, we recover the Black and Scholes model

$$\frac{dS_t}{S_t} = \mu dt + \sigma d\beta_t$$

with

$$\mu = \sigma \frac{m}{T}.$$

In this special and interesting case, all the computations can be made explicitly.

Proposition 13. *Let*

$$\delta = \frac{s^2 - T}{T}$$

the relative variance ($\delta = 0$ corresponds to the Black and Scholes model). The optimal proportion processes Π_t and the optimal expected utilities $u\,(x, \nu)$ for the utility functions U below are given as follows:

1. Logarithmic utility $U :]0, +\infty[\to \mathbb{R}$, $x \to \ln x$.

$$\Pi_t = \frac{1}{\sigma} \frac{\delta B_t + m}{\delta t + T} \, , \, 0 \leq t \leq T,$$

$$u(x, \nu) = \ln x + \frac{1}{2} \left(\delta - \ln (1 + \delta) + \frac{m^2}{T} \right).$$

2. Power utility $U :]0, +\infty[\to \mathbb{R}$, $x \to \frac{x^\alpha}{\alpha}$, $\alpha \in]0, 1[$.

$$\Pi_t = \frac{1}{\sigma} \frac{\delta B_t + m}{\delta t + T(1 - \alpha) - \alpha \delta T} \, , \, 0 \leq t \leq T,$$

$$u(x, \nu) = \frac{x^\alpha}{\alpha} \frac{1}{\sqrt{1 + \delta}} \left(\frac{1 - \alpha}{\frac{1}{1+\delta} - \alpha} \right)^{\frac{1-\alpha}{2}} \exp \left(\frac{\alpha m^2}{2(T(1 - \alpha) - \alpha \delta T)} \right).$$

5 Modeling of a Weak Information Flow

In this last section, we give a framework to model a weak information flow. Precisely, we study three cases:

1. The insider has a knowledge of all the conditional laws $\mathbb{P}(Y \in dy \mid \mathcal{F}_t)$, $0 \leq t < T$.
2. The insider is allowed to update his weak information according to the information he receives.
3. The insider is in the following position: At time t, he receives a weak information about the price S_{t+dt}, this anticipation being only valid for the infinitesimal time dt and at $t + dt$ the investor receives some new information and makes a new anticipation, and so on....

5.1 Dynamic Conditioning

In this section, we consider an insider who knows all the conditional laws of Y. Precisely, with Y we associate a continuous $\mathcal{F}$-adapted process $(\nu_t)_{0 \leq t \leq T}$ of probability measures on $\mathcal{P}$ (assumed to be the conditional laws of Y under the effective probability of the market) such that $\nu_T = \delta_Y$.

We will assume that for $0 \leq t < T$, ν_t admits an almost surely strictly positive bounded density ξ_t with respect to $\mathbb{P}(Y \in dy \mid \mathcal{F}_t)$, i.e.

$$\nu_t(dy) = \xi_t(y) \mathbb{P}(Y \in dy \mid \mathcal{F}_t) \, .$$

Remark 26. The terminal condition $\nu_T = \delta_Y$ is equivalent to $\xi_T = 1$.

Let $\mathcal{E}^\nu$ be the set of probability measures $\mathbb{Q}$ on Ω such that:

1. $\mathbb{Q}$ is equivalent to $\mathbb{P}$

2. $\mathbb{Q}(Y \in dy \mid \mathcal{F}_t) = \nu_t(dy), t < T$.

In order to ensure that $\mathcal{E}^\nu$ is non empty we have to make the following additional assumption on ν_t.

Assumption 14 *There exists a $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathcal{P})$ measurable process $(\lambda_t^y)_{0 \le t < T}$ and an adapted d-dimensional semimartingale $(A_t)_{0 \le t < T}$ such that:*

1. *For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$ and for $0 \le t < T, 1 \le i, j \le d$*

$$\mathbb{E}\left(\int_0^t \left(\lambda_u^{y,i}\right)^2 d\langle A^i, A^j\rangle_u\right) < +\infty$$

2.

$$\nu_t(dy) = \nu_0(dy) + \int_0^t \left(\lambda_s^y \mathbb{P}_Y(dy)\right) dA_s,\ t < T$$

3.

$$\nu_t \to_{t \to T}^{weakly} \delta_Y.$$

In order to understand this, let us give a simple example of a sequence $(\nu_t)_{0 \le t < T}$ which satisfies this assumption. Assume that $(S_t)_{0 \le t \le T}$ is a d-dimensional standard Brownian motion and that

$$\nu_t(dy) = q_{T-t}(S_t, y)dy$$

where $(q_t)_{0 \le t < T}$ is the transition function of the diffusion with generator

$$\mathcal{L} = b(x)\nabla + \frac{1}{2}\Delta$$

where b is a bounded smooth function whose all partial derivatives are also bounded.

In this case, it easily seen (by Itô's formula) that our assumption holds with

$$\lambda_u^y = \sqrt{2\pi T}\, e^{\frac{y^2}{2T}}\, \frac{\partial q_{T-u}}{\partial x}(S_u, y)$$

and

$$A_t = S_t - \int_0^t b(S_u)du.$$

Notice that in this case

$$\mathcal{E}^\nu = \{\mathbb{Q}\}$$

where $\mathbb{Q}$ is the probability equivalent to $\mathbb{P}$ such that

$$S_t - \int_0^t b(S_u)du$$

is a martingale under $\mathbb{Q}$.

The following proposition characterizes the set $\mathcal{E}^\nu$. Before we state it, recall that a probability measure valued process $(\nu_t)_{0 \le t < T}$ is called a martingale, if for any bounded and measurable function f, the process $\int_{\mathcal{P}} f(y)\nu_t(dy)$ is a martingale.

Proposition 15. *Let $\mathbb{Q} =D\ \mathbb{P}$ a probability measure on Ω equivalent to $\mathbb{P}$, then the following assertions are equivalent:*

i) $\mathbb{Q} \in\mathcal{E}^{\nu}$

ii) The measure valued process $(\nu_t)_{0\leq t\leq T}$ is a $\mathcal{F}$ martingale under $\mathbb{Q}$

iii) The process $(D_t\xi_t(Y))_{0\leq t\leq T}$ is a $\mathcal{G}$ martingale under $\mathbb{P}$, where $D_t = \mathbb{E}(D \mid \mathcal{F}_t)$ and where $\mathcal{G}$ is the initial enlargement of $\mathcal{F}$ by Y

iv) The process $(A_t)_{0\leq t<T}$ is a $\mathcal{F}$ martingale under $\mathbb{Q}$.

Proof. i) $\Rightarrow$ ii)

Let us consider $\mathbb{Q} \in\mathcal{E}^{\nu}$.

Then for all bounded and measurable function f

$$\mathbb{E}^{\mathbb{Q}}\left(f(Y) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} f(y)\,\nu_t(dy)\,,\ t\leq T,$$

which implies immediately ii).

ii) $\Rightarrow$ iii)

If $(\nu_t)_{0\leq t\leq T}$ is a $\mathcal{F}-$martingale under $\mathbb{Q}$ then

$$\mathbb{E}\left(D \mid \mathcal{G}_t\right) = \xi_t(Y)\,\mathbb{E}\left(D \mid \mathcal{F}_t\right),\ t\leq T.$$

Indeed, for all bounded and measurable function f,

$$\mathbb{E}^{\mathbb{Q}}\left(f(Y) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} f(y)\,\nu_t(dy)\,,\ t\leq T$$

thus,

$$\mathbb{E}\left(\frac{D}{D_t}\,f(Y) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} f(y)\,\xi_t(y)\,\mathbb{P}\left(Y\in dy \mid \mathcal{F}_t\right),\ t\leq T$$

and so,

$$\mathbb{E}\left(\frac{D}{D_t}\,f(Y) \mid \mathcal{F}_t\right) = \mathbb{E}\left(f(Y)\,\xi_t(Y) \mid \mathcal{F}_t\right),\ t\leq T.$$

This means that for all bounded and $\mathcal{F}_t-$ measurable functional F

$$\mathbb{E}\left(\frac{D}{D_t}\,f(Y)\,F\right) = \mathbb{E}\left(f(Y)\,\xi_t(Y)\,F\right),\ t\leq T$$

which provides

$$\mathbb{E}\left(D \mid \mathcal{G}_t\right) = \xi_t(Y)\,D_t,\ t\leq T,$$

so that the process $(D_t\xi_t(Y))_{0\leq t\leq T}$ is a $\mathcal{G}$ martingale under $\mathbb{P}$.

iii) $\Rightarrow$ i)

In this case, we have

$$\mathbb{E}\left(D \mid \mathcal{G}_t\right) = \xi_t(Y)\,\mathbb{E}\left(D \mid \mathcal{F}_t\right),\ t\leq T$$

which implies that for all bounded and measurable function f,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\left(f\left(Y\right) \mid \mathcal{F}_t\right) &= \mathbb{E}\left(f\left(Y\right)\frac{\mathbb{E}(D|\mathcal{G}_t)}{D_t} \mid \mathcal{F}_t\right) \\ &= \mathbb{E}\left(f\left(Y\right)\xi_t\left(Y\right) \mid \mathcal{F}_t\right) \\ &= \int_{\mathcal{P}} f\left(y\right)\xi_t\left(y\right)\mathbb{P}\left(Y \in dy \mid \mathcal{F}_t\right) \\ &= \int_{\mathcal{P}} f\left(y\right)\nu_t\left(dy\right).\end{aligned}$$

iv)⇔ii)
Immediate. □

The set $\mathcal{E}^{\nu}$ is hence the set of martingale measures for A.

5.2 Dynamic Correction of a Weak Information

A Useful Convergence Lemma for SDE's

Before, we define our framework for a dynamic correction of the weak information, we state a lemma about the continuity (with respect to a suitable norm) of solutions of stochastic differential equations with respect to the drift. This lemma will be used in the following.

Let us consider a sequence of Borel functions

$$a_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d,\ n \in \mathbb{N}$$

and let us also consider a Borel function $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}_d\left(\mathbb{R}\right)$, where $\mathcal{M}_d\left(\mathbb{R}\right)$ is the space of $d \times d$ matrix.

We make the following assumptions. There exist non negative constants K_1 and K_2 such that for $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$\left\|a_n\left(t,x\right) - a_n\left(t,y\right)\right\| + \left\|\sigma\left(t,x\right) - \sigma\left(t,y\right)\right\| \leq K_1\left\|x-y\right\|$$

and

$$\left\|a_n\left(t,x\right)\right\|^2 + \left\|\sigma\left(t,x\right)\right\|^2 \leq K_2\left(1+\left\|x\right\|^2\right).$$

Furthermore, we assume that the following pointwise convergence holds

$$a_n \to_{n\to+\infty} a$$

where $a : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a Borel function. Notice that we do not assume the continuity with respect to t of the functions a_n.

Now we consider on a filtered probability space

$$\left(\Omega, \left(\mathcal{F}_t\right)_{0\leq t\leq T}, \mathbb{P}\right)$$

which satisfies the usual conditions and on which a standard d-dimensional Brownian motion is defined, the sequence X^n, where $\left(X^n_t\right)_{0\leq t\leq T}$ is the solution of the stochastic differential equation

$$X_t^n = x_0 + \int_0^t a_n\left(s, X_s^n\right) ds + \int_0^t \sigma\left(s, X_s^n\right) dW_s.$$

We can note that the assumptions made on a_n and σ ensure the existence and the uniqueness of such a solution. We have the following theorem.

Theorem 21. *Under the above conditions, we have*

$$\lim_{n \to +\infty} \mathbb{E}\left(\sup_{0 \le t \le T} \left\| X_t^n - X_t \right\|^2\right) = 0$$

where $(X_t)_{0 \le t \le T}$ *is the solution of the stochastic differential equation*

$$X_t = x_0 + \int_0^t a\left(s, X_s\right) ds + \int_0^t \sigma\left(s, X_s\right) dW_s.$$

Proof. For notational convenience, we make the proof in dimension $d = 1$ but it immediately extends to the d dimensional case. Since for $x, y \in \mathbb{R}^2$

$$(x+y)^2 \le 2\left(x^2 + y^2\right) \tag{30}$$

we have for $t \in [0, T]$ and $n \in \mathbb{N}$

$$\begin{aligned} \left(X_t^n - X_t\right)^2 \le \quad & 2\left[\int_0^t \left(a_n\left(u, X_u^n\right) - a\left(u, X_u\right)\right) du\right]^2 \\ & +2\left[\int_0^t \left(\sigma\left(u, X_u^n\right) - \sigma\left(u, X_u\right)\right) dW_u\right]^2. \end{aligned}$$

Now, from Cauchy-Schwarz inequality and (30)

$$\begin{aligned} \left[\int_0^t \left(a_n\left(u, X_u^n\right) - a\left(u, X_u\right)\right) du\right]^2 \le\ & 2T \int_0^t \left(a_n\left(u, X_u^n\right) - a_n\left(u, X_u\right)\right)^2 du \\ & +2T \int_0^t \left(a_n\left(u, X_u\right) - a\left(u, X_u\right)\right)^2 du. \end{aligned}$$

Thus,

$$\begin{aligned} \left[\int_0^t \left(a_n\left(u, X_u^n\right) - a\left(u, X_u\right)\right) du\right]^2 \le\ & 2TK_1^2 \int_0^t \left(X_u^n - X_u\right)^2 du \\ & +2T \int_0^t \left(a_n\left(u, X_u\right) - a\left(u, X_u\right)\right)^2 du. \end{aligned}$$

On the other hand, from Burkholder-Davis-Gundy inequality, for $0 \le t \le \tau$

$$\begin{aligned} & \mathbb{E}\left(\sup_{0 \le t \le \tau} \left[\int_0^t \left(\sigma\left(u, X_u^n\right) - \sigma\left(u, X_u\right)\right) dW_u\right]^2\right) \\ & \quad \le 4K_1^2 \mathbb{E}\left(\int_0^\tau \left(X_u^n - X_u\right)^2 du\right). \end{aligned}$$

Putting things together, we deduce the following estimation

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left(X_t^n-X_t\right)^2\right)\leq(2T+8)\,K_1^2\mathbb{E}\left(\int_0^\tau\left(X_u^n-X_u\right)^2du\right)\\ +2T\mathbb{E}\left(\int_0^t\left(a_n\left(u,X_u\right)-a\left(u,X_u\right)\right)^2du\right).\end{aligned}\tag{31}$$

We apply now Gronwall's lemma to obtain

$$\begin{aligned}\mathbb{E}\left(\left(X_\tau^n-X_\tau\right)^2\right)\leq\ & 2T\,(2T+8)\,K_1^2e^{(2T+8)K_1^2\tau}\int_0^\tau e^{-(2T+8)K_1^2u}G_n\left(u\right)du\\ & +2TG_n\left(\tau\right)\end{aligned}$$

where

$$G_n\left(\tau\right):=\mathbb{E}\left(\int_0^\tau\left(a_n\left(u,X_u\right)-a\left(u,X_u\right)\right)^2du\right).$$

The uniform linear growth assumption on a_n allows to use the dominated convergence theorem which ensures first the following pointwise convergence

$$G_n\to_{n\to+\infty}0$$

A new use of the dominated convergence theorem in (31) gives the expected result, i.e. for $0\leq\tau\leq T$

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left(X_t^n-X_t\right)^2\right)\to_{n\to+\infty}0$$

This complete the proof. □

Dynamic Correction of the Weak Information Flow

In this section, we define a framework for a dynamic correction of the weak information at each time t. The insider is allowed to update his weak information according to the information he receives. The reasoning here is more pathwise oriented in the spirit of the notion of minimal model (see subsection 3.1).

Here we shall assume that the dynamics of $(S_t)_{0\leq t\leq T}$ under the martingale measure $\mathbb{P}$ are given by

$$S_t=s_0+\int_0^t\operatorname{diag}\left(S_u\right)\sigma\left(S_u\right)dW_u,\qquad 0\leq t\leq T$$

where $(W_t)_{0\leq t\leq T}$ is a d-dimensional standard Brownian motion, $s_0\in\left(\mathbb{R}_+^*\right)^d$ and σ a positive definite symmetric bounded C^∞ function with bounded partial derivatives function satisfying

$$\inf_{x\in\mathbb{R}^d}\ \|\sigma\sigma^*\left(x\right)\|\geq a>0.$$

Let us consider a sequence of subdivisions

$$\mathcal{S}_n=\{0\leq t_0<...<t_i<...<t_n=T\},\quad n\in\mathbb{N}^*$$

of the time interval $[0,T]$, whose mesh tends to 0 when $n\to+\infty$.

The idea now is to associate with S_T an $\sigma(S_{t_i})$–adapted random sequence $(\nu_{t_i})_{i=0,\dots,n-1}$ of probability measures on $\mathcal{P}$ corresponding to an updating of the weak information on S_T at time t_i. This updating can come from the observation of the prices in the time interval $[0,t_i]$ as well as the learning of a new information on S_T. Let us now try to construct a model for the price process which takes into account these updatings. We shall furthermore assume that the insider has no other information on the price process.

At time t_i the insider "learns" ν_{t_i} (which **erases completely** $\nu_{t_{i-1}}$) and constructs a probabilistic bridge in the time interval $[t_i,T]$ which condition S_T to follow conditionally to the past filtration $\mathcal{F}_{t_i}$ the law ν_{t_i}. Since this anticipation is only valid in the time interval $[t_i,t_{i+1})$, according to Theorem 15 and Remark 18, the model is given by

$$\begin{aligned} S_u - S_{t_i} &= \int_{t_i}^{u} \left(\tilde{\sigma}^*\tilde{\sigma}\right)(S_v) \frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)} \nu_{t_i}(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)} \nu_{t_i}(dy)} dv \\ &+ \int_{t_i}^{u} \tilde{\sigma}(S_v)\, dW_v, t_i \le u < t_{i+1} \end{aligned}$$

where $(W_t)_{0\le t\le T}$ is a d-dimensional standard Brownian motion and $\tilde{\sigma}(x) = \operatorname{diag}(x)\,\sigma(x)$.

Putting things together, we obtain the following dynamics

$$S_t = s_0 + \int_0^t \left(\tilde{\sigma}^*\tilde{\sigma}\right)(S_v)\, a_n(v,S_v)\, dv + \int_0^t \tilde{\sigma}(S_v)\, dW_v, 0 \le t \le T$$

where

$$a_n(v,S_v) = \sum_{i=0}^{n-1} \left(\frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)} \nu_{t_i}(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)} \nu_{t_i}(dy)} \right) 1_{[t_i,t_{i+1})}(v).$$

Notice now that the following pointwise convergence holds

$$a_n \to_{n\to+\infty} a$$

where

$$a_n(t,x) = \int_{\mathbb{R}^d} \frac{\nabla p_{T-t}(S_t,y)}{p_{T-t}(S_t,y)} \nu_t(dy).$$

Hence, if we are under the assumptions of Theorem 21, then

$$\begin{aligned} S_t &= s_0 + \int_0^t \left(\tilde{\sigma}^*\tilde{\sigma}\right)(S_u) \int_{\mathbb{R}^d} \frac{\nabla p_{T-u}}{p_{T-u}}(S_u,y)\,\nu_u(dy)\, du \\ &+ \int_0^t \tilde{\sigma}(S_u)\, dW_u, 0 \le t \le T \end{aligned} \tag{32}$$

is a model for the price process which takes into account the dynamic correction of the weak information flow.

An example of Dynamic Correction

Let us now assume that the dynamics of $(S_t)_{0\leq t\leq T}$ under $\mathbb{P}$ is given by

$$dS_t = \sigma S_t dB_t, 0 \leq t \leq T$$

where $(B_t)_{0\leq t\leq T}$ is a standard Brownian motion whose filtration is $\mathcal{F}$ and σ a strictly positive constant. With this model, we associated the two parameter Gaussian diffusion

$$dB_t = \frac{\left(s^2 - T\right) B_t + mT}{\left(s^2 - T\right) t + T^2} dt + d\beta_t \tag{33}$$

m being a mean parameter and s a variance parameter (we recall that $B_T \sim \mathcal{N}\left(m, s^2\right)$). This model is the model which has been introduced at the end of section 4.2.

With this two parameters model, it is natural to make the maximum likelihood method on the time interval $[0,t]$, $t > 0$, in order to check our anticipation. This gives after some straightforward computations the following estimators

$$m_t = \frac{T}{t} B_t,$$

and

$$s_t^2 = 0.$$

Hence, we apply formally the dynamic correction procedure, we obtain thanks to (32)

$$dB_t = \frac{B_t}{t} dt + dW_t$$

which is a singular equation (Precisely, if B is a standard Brownian motion then the natural filtration of

$$\left(B_t - \int_0^t \frac{B_s}{s} ds\right)_{t<T}$$

completed by $\sigma\left(B_T\right)$ is the natural filtration of $(B_t)_{0\leq t\leq T}$, see [38] for further details). This helps understand, that without any exogenous information on B_T (in this case ν_t is constructed only by observing the past of a Brownian motion), the dynamic correction leads to a singularity.

Nevertheless, for a general process $(\nu_t)_{0\leq t<T}$, the dynamic correction procedure applied to the model (33) , gives according to 32

$$dB_t = \frac{\int_{\mathbb{R}} y\, \nu_t\left(dy\right) - B_t}{T - t} dt + d\tilde{W}_t.$$

This equation implies something interesting, indeed it implies

$$\mathbb{E}\left(B_t\right) = (T - t) \int_0^t \frac{\mathbb{E}\left(\mu_s\right)}{(T - s)^2} ds,\ t < T$$

with $\mu_t = \int_{\mathbb{R}} y\, \nu_t\left(dy\right)$. This corresponds to the natural intuition that at each time $t < T$, the tangent to the curve $s \to \mathbb{E}\left(B_s\right)$ hits the line $s = T$ at the point $\left(T, \mathbb{E}^*\left(\mu_t\right)\right)$, a well-known phenomenon in physics which is related to the notion of caustic.

5.3 Dynamic Information Arrival

Now, we would like to present in this short section a nice construction. We will not be rigorous in order to make understand the main intuitions.

Here again, we consider the case where S is one-dimensional and assume furthermore that there exist a bounded C^∞ function $\sigma : \mathbb{R}_+ \to [a, +\infty[$ $(a > 0)$ and a $\mathbb{P}$-standard Brownian motion $(B_t)_{0 \le t \le T}$ whose filtration is $\mathcal{F}$ such that

$$dS_t = S_t \sigma(S_t)\, dB_t, \quad 0 \le t \le T.$$

Let us consider an insider who is in the following position: At time t, he receives a weak information about the price S_{t+dt}, this anticipation being only valid for the infinitesimal time dt and at $t + dt$ the investor receives some new information and makes a new anticipation, and so on....

Such an insider tries hence to construct a probability measure $\mathbb{P}^*$ on Ω such that

$$\mathbb{P}^*(S_{t+dt} \in dy \mid \mathcal{F}_t) = \xi_t(y)\, \mathbb{P}(S_{t+dt} \in dy \mid \mathcal{F}_t)$$

$y \to \xi_t(y)$ being the $\mathcal{F}_t-$measurable function, corresponding to the weak information that the investor receives at time t.

A canonical way to construct $\mathbb{P}^*$ is the following: At time t, he learns the weak information (S_{t+dt}, ξ_t) and constructs in the sense of Section 2 a probabilistic bridge in the time interval $[t, t+dt)$ which forces S_{t+dt} to follow conditionally to the past filtration $\mathcal{F}_t$ (which is not trivial contrary to the static case) the law $\nu_t(dy) = \xi_t(y)\, \mathbb{P}(S_{t+dt} \in dy \mid \mathcal{F}_t)$. Let us see what it implies on $\mathbb{P}^*$.

Let us denote by $\widetilde{\sigma}$ the function $x \to x\sigma(x)$. It is easily seen, from a representation theorem, that we have

$$\xi_t(S_{t+dt}) = 1 + \xi_t'(S_t)\, \widetilde{\sigma}(S_t)\, dB_t$$

(recall that it is assumed that ξ_t is differentiable) because of the normalization

$$\mathbb{E}(\xi_t(S_{t+dt}) \mid \mathcal{F}_t) = 1$$

Hence, for all test function f, we must have thanks to Itô's formula (notice that $\xi_t(S_t) = 1$)

$$\begin{aligned}\mathbb{E}^*(f(S_{t+dt}) \mid \mathcal{F}_t) &= \mathbb{E}(\xi_t(S_{t+dt})\, f(S_{t+dt}) \mid \mathcal{F}_t) \\ &= f(S_t) + f'(S_t)\, \xi_t'(S_t)\, \widetilde{\sigma}(S_t)^2\, dt + \tfrac{1}{2} f''(S_t)\, \widetilde{\sigma}^2(S_t)\, dt\end{aligned}$$

which implies that the semimartingale decomposition of S under $\mathbb{P}^*$ is

$$dS_t = \xi_t'(S_t)\, \widetilde{\sigma}(S_t)^2\, dt + \widetilde{\sigma}(S_t)\, d\beta_t$$

where β is a standard Brownian motion under $\mathbb{P}^*$.

6 Comments

It seems that there are many advantages to the use of the weak approach instead of the strong one in the modeling of information on financial markets. The first reason is the robustness of this kind of modeling. Mathematically, it means that the map $\nu \to \mathbb{P}^{\nu}$ is continuous in the weak topology (as soon as we have a continuous version for the maps $y \to \mathbb{P}(A \mid Y = y)$). In practice, it means that a little modification on the parameters of the model does not change completely the model. Of course this robustness does not hold for initial enlargement of filtration. The second reason is practical. Indeed, the weak approach generates simple models which are easy to calibrate and to implement numerically. And finally, last but not least, the third reason is theoretical. Indeed, it has been seen that the theory of initial enlargement of filtration can be deduced from the weak approach by taking formally for ν some anticipative measures (precisely $\nu = \delta_Y$ gives the enlargement $\mathcal{F} \vee \sigma(Y)$).

To conclude, it is important to stress the fact that more works need to be done.

1. On one hand, it would be very interesting to study more thoroughly some examples of incomplete markets, such as stochastic volatility models.
2. On the other hand, it would also be interesting to apply the present approach to equilibrium price models in the spirit of [4] and [35]; for instance, given two agents acting in the same market S with different weak anticipations, what is the equilibrium price of S?

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Duality in constrained optimal investment and consumption problems: a synthesis

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1 Dual Problems Made Easy

These lectures are all about optimal investment/consumption problems, usually with some 'imperfection', such as transaction costs, or constraints on the permitted portfolios, or different interest rates for borrowing and lending, or margin requirements for borrowing, or even just incomplete markets. Some time ago, Karatzas, Lehoczky and Shreve (1987), and Cox and Huang (1989) realised that the use of duality methods provided powerful insights into the solutions of such problems, using them to prove the form of the optimal solution to significant generalisations of the original Merton (1969) problem, which Merton had proved using (very problem-specific) optimal control methods. These duality methods have become very popular in the intervening years, and now it seems that when faced with one of these problems, the steps are:

(i) try to solve the problem explicitly;

(ii) if that fails, find the dual form of the problem;

(iii) try to solve the dual problem;

(iv) if that fails, assume that investors have log utilities and try (iii) again;

(v) if that still fails, generalize the problem out of recognition and redo (ii);

(vi) write a long and technical paper, and submit to a serious journal.

As so often happens, when all the details are written up, it can be very hard to see the main lines of what is going on, and I have to say now that with most of the papers I have read in the literature, I find it easier to take the statement of the problem and

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work out the dual form for myself than to try to follow the arguments presented in the papers. This is not to say that the paper is redundant, but rather that one can very quickly get to the form of the dual problem, even if proving that the dual and primal problems have equal value remains a substantial task.

There is in fact a unified (and very simple) approach to find the dual form of the problem which works in a wide range of cases[1]. We can think of this as the Pontryagin approach to dynamic programming; or we can think of it in the 'Hamiltonian' language of Bismut (1973), (1975) (see, for example, Malliaris and Brock (1982) for the outline of the method; Chow (1997) also emphasises the efficacy of this approach). To illustrate what I mean, let me now present the method applied to the very simplest example.

Example 0. Suppose we consider an investor who may invest in any of n stocks and in a riskless bank account generating interest at rate r_t. Then the wealth process X of the investor satisfies the dynamics

$$dX_t = r_t X_t dt + \theta_t(\sigma_t dW_t + (b_t - r_t \mathbf{1})dt), \quad X_0 = x, \tag{1}$$

where all processes are adapted to the filtration of the standard d-dimensional Brownian motion W, σ takes values in the set of $n \times d$ matrices, and all other processes have the dimensions implied by (1)[2]. The process θ is the vector of amounts of wealth invested in each of the stocks. The investor aims to find

$$\sup \mathbb{E}\Big[\, U(X_T) \,\Big], \tag{2}$$

where $T > 0$ is a fixed time-horizon, and the function $U(\cdot)$ is strictly increasing, strictly concave, and satisfies the Inada conditions[3].

Now we view the dynamics (1) of X as some *constraint* to be satisfied by X, and we turn the constrained optimisation problem (2) into an *unconstrained* optimisation problem by introducing appropriate Lagrange multipliers. To do this, let the positive process Y satisfy[4]

$$dY_t = Y_t(\beta_t dW_t + \alpha_t dt), \tag{3}$$

and consider the integral $\int_0^T Y_s dX_s$. On the one hand, integration by parts gives

$$\int_0^T Y_s dX_s = X_T Y_T - X_0 Y_0 - \int_0^T X_s dY_S - [X, Y]_T, \tag{4}$$

and on the other we have (provided constraint/dynamic (1) holds)

1 ... but see Section 6.

2 So, for example, $\mathbf{1}$ is the column n-vector all of whose entries are 1.

3 $\lim_{x\downarrow 0} U'(x) = \infty$, $\lim_{x\uparrow\infty} U'(x) = 0$. For concreteness, we are assuming that X must remain non-negative.

4 It is not necessary to express Y in exponential form, but it turns out to be more convenient to do this in any example where processes are constrained to be non-negative.

$$\int_0^T Y_s dX_s = \int_0^T Y_s \theta_t \sigma_s dW_s + \int_0^T Y_s \{r_s X_s + \theta_s (b_s - r_s \mathbf{1})\} ds. \tag{5}$$

Assuming that expectations of stochastic integrals with respect to W vanish, the expectation of $\int_0^T Y_s dX_s$ is from (4)

$$\mathbb{E}\Big[X_T Y_T - X_0 Y_0 - \int_0^T Y_s \{\alpha_s X_s + \theta_s \sigma_s \beta_s\} ds\Big], \tag{6}$$

and from (5)

$$\mathbb{E}\Big[\int_0^T Y_s \{r_s X_s + \theta_s (b_s - r_s \mathbf{1})\} ds\Big]. \tag{7}$$

Since these two expressions must be equal for any feasible X, we have that the Lagrangian

$$\begin{aligned} \Lambda(Y) \equiv \sup_{X \geq 0, \theta} \mathbb{E}\Big[U(X_T) &+ \int_0^t Y_s \{r_s X_s + \theta_s (b_s - r_s \mathbf{1})\} ds \\ &- X_T Y_T + X_0 Y_0 + \int_0^T Y_s \{\alpha_s X_s + \theta_s \sigma_s \beta_s\} ds\Big] \\ = \sup_{X \geq 0, \theta} \mathbb{E}\Big[U(X_T) &- X_T Y_T + X_0 Y_0 \\ &+ \int_0^T Y_s \{r_s X_s + \theta_s (b_s - r_s \mathbf{1}) + \alpha_s X_s + \theta_s \sigma_s \beta_s\} ds\Big] \end{aligned} \tag{8}$$

is an upper bound for the value (2) whatever Y we take, and will hopefully be equal to it if we minimise over Y.

Now the maximisation of (8) over $X_T \geq 0$ is very easy; we obtain

$$\begin{aligned} \Lambda(Y) = \sup_{X \geq 0, \theta} \mathbb{E}\Big[\tilde{U}(Y_T) &+ X_0 Y_0 \\ &+ \int_0^T Y_s \{r_s X_s + \theta_s (b_s - r_s \mathbf{1}) + \alpha_s X_s + \theta_s \sigma_s \beta_s\} ds\Big], \end{aligned}$$

where $\tilde{U}(y) \equiv \sup_x [U(x) - xy]$ is the convex dual of U. The maximisation over $X_s \geq 0$ results in a finite value if and only if the complementary slackness condition

$$r_s + \alpha_s \leq 0 \tag{9}$$

holds, and maximisation over θ_s results in a finite value if and only if the complementary slackness condition

$$\sigma_s \beta_s + b_s - r_s \mathbf{1} = 0 \tag{10}$$

holds. The maximised value is then

$$\Lambda(Y) = \mathbb{E}\Big[\tilde{U}(Y_T) + X_0 Y_0\Big]. \tag{11}$$

The dual problem therefore ought to be

$$\inf_Y \Lambda(Y) = \inf_Y \mathbb{E}\Big[\tilde{U}(Y_T) + X_0 Y_0\Big], \tag{12}$$

where Y is a positive process given by (3), where α and β are understood to satisfy the complementary slackness conditions (9) and (10). In fact, since the dual function $\tilde{U}(\cdot)$ is decreasing, a little thought shows that we want Y to be big, so that the 'discount rate' α will be as large as it can be, that is, the inequality (9) will actually hold with equality.

We can interpret the multiplier process Y, now written as

$$Y_t = Y_0 \exp\{-\int_0^t r_s ds\}.Z_t,$$

as the product of the initial value Y_0, the riskless discounting term $\exp(-\int_0^t r_s ds)$, and a (change-of-measure) martingale Z, whose effect is to convert the rates of return of all of the stocks into the riskless rate. In the case where $n = d$ and σ has bounded inverse, we find the familiar result of Karatzas, Lehoczky and Shreve (1987), for example, that the marginal utility of optimal wealth is the pricing kernel, or state-price density.

The informal argument just given leads quickly to a candidate for the dual problem; to summarise, the key elements of the approach are:

(a) *write down the dynamics;*

(b) *introduce a Lagrangian semimartingale Y, often in exponential form;*

(c) *transform the dynamics using integration-by-parts;*

(d) *assemble the Lagrangian, and by inspection find the maximum, along with any dual feasibility and complementary slackness conditions.*

We shall see this approach used repeatedly through these lectures; it is a mechanistic way of discovering the dual problem of any given primal problem.

How close to a proof is the argument just given? At first sight, there seem to be big gaps, particularly in the assumption that means of stochastic integrals with respect to local martingales should be zero. But on the other hand, we are looking at a Lagrangian problem, and provided we can *guess* the optimal solution, we we should be able then to *verify* it using little more than the fact that a concave function is bounded above by any supporting hyperplane. So the argument would go that if we have optimal X^*, we would define the dual variable $Y_T^* = U'(X_T^*)$, and simply confirm that the Lagrangian with this choice of Y is maximised at X^*. However, there are problems with this; firstly, we do not know that the supremum is a maximum, and we may have to build X^* as a limit (in what topology?) of approximating X^n; secondly, as we shall soon see in a more general example, the marginal utility of optimal wealth is not necessarily a state-price density. The simple heuristic given above comes tantalisingly close to being a proof of what we want; we can see it, but we are in fact still separated from it by a deep chasm. Getting across this still requires significant effort, though later in Section 3 we shall build a bridge to allow us to cross the chasm - though even this may not be easy to cross.

2 Dual Problems Made Concrete

Here are some further examples to get practice on.

Example 1. The investor of Example 0 now consumes from his wealth at rate c_t at time t, so that the dynamics of his wealth process becomes

$$dX_t = r_t X_t dt + \theta_t(\sigma_t dW_t + (b_t - r_t\mathbf{1})dt) - c_t dt, \quad X_0 = x, \tag{13}$$

His objective now is to find

$$\sup \mathbb{E}\Big[\int_0^T U(s, c_s)ds + U(T, X_T) \Big], \tag{14}$$

where $T > 0$ is a fixed time-horizon, and for each $0 \le s \le T$ the function $U(s, \cdot)$ is strictly increasing, strictly concave, and satisfies the Inada conditions.

EXERCISE 1. Apply the general approach given above to show that the dual problem is to find

$$\inf_Y \Lambda(Y) \equiv \inf_Y \mathbb{E}\Big[\int_0^T \tilde{U}(s, Y_s)ds + \tilde{U}(T, Y_T) + X_0 Y_0\Big], \tag{15}$$

where Y is a positive process satisfying

$$dY_t = Y_t(\beta_t dW_t + \alpha_t dt), \tag{16}$$

with

$$\alpha_t = -r_t, \quad \sigma_s\beta_s + b_s - r_s\mathbf{1} = 0 \tag{17}$$

Example 2. (El Karoui and Quenez (1995)). In this problem, the agent's wealth once again obeys the dynamics (13), but now the objective is to find the *super-replication price*, that is, the smallest value of x such that by judicious choice of θ and c he can ensure that

$$X_T \ge B \quad \text{a.s.},$$

where B is some given $\mathcal{F}_T$-measurable random variable.

EXERCISE 2. Replacing the objective (2) from Example 0 by

$$\sup \mathbb{E}\big[u_0(X_T - B)\big],$$

where $u_0(x) = -\infty$ if $x < 0$, $u_0(x) = 0$ if $x \geq 0$, show that the super-replication price is

$$\sup_{\beta \in \mathcal{B}_0} \mathbb{E}[BY_T(\beta)], \tag{18}$$

where $Y_t(\beta)$ is the solution to

$$dY_t = Y_t(-r_t dt + \beta_t dW_t), \quad Y_0 = 1,$$

and $\mathcal{B}_0 \equiv \{\text{adapted } \beta \text{ such that } \sigma_s \beta_s + b_s - r_s \equiv 0\}$.

Remark 1. This example shows that we need in general to allow the utility to depend on ω. The super-replication price (18) can equally be expressed as

$$\sup_{\mathbb{Q} \in \mathcal{M}} \mathbb{E}^{\mathbb{Q}}[\exp(-\int_0^T r_s ds) B],$$

where $\mathcal{M}$ denotes the set of equivalent martingale measures.

Example 3. (Kramkov and Schachermayer (1999)). This example is the general form of Example 0. In this situation, the asset price processes S are general non-negative semimartingales, and the attainable wealths are random variables X_T which can be expressed in the form

$$X_T = x + \int_0^T H_u dS_u$$

for some previsible H such that the process $X_t \equiv x + \int_0^t H_u dS_u$ remains non-negative.

EXERCISE 3A. If $\mathcal{X}(x)$ denotes the set of such random variables X_T, and if $\mathcal{Y}$ denotes the set of all positive processes Y such that $Y_t X_t$ is a supermartingale for all $X \in \mathcal{X}(x)$, show that the dual form of the problem is

$$\sup_{X_T \in \mathcal{X}(x)} \mathbb{E}[U(X_T)] = \inf_{Y \in \mathcal{Y}} \mathbb{E}[\tilde{U}(Y_T) + xY_0]. \tag{19}$$

EXERCISE 3B. (based on Example 5.1 bis of Kramkov and Schachermayer (1999)). Consider a two-period model where the price process is (S_0, S_1), with $S_0 \equiv 1$, and S_1 taking one of a sequence $(x_n)_{n \geq 0}$ of values decreasing to zero, with probabilities $(p_n)_{n \geq 0}$. Suppose that $x_0 = 2$, $x_1 = 1$, and suppose that

$$\frac{p_0}{\sqrt{2}} > \sum_{n \geq 1} \frac{p_n(1 - x_n)}{\sqrt{x_n}}. \tag{20}$$

The agent has utility $U(x) = \sqrt{x}$, initial wealth 1, and his portfolio choice consists simply of choosing the number $\lambda \in [-1, 1]$ of shares to be held. If he holds λ, his expected utility is

$$\mathbb{E}U(X_1) = \sum_{n \geq 0} p_n \sqrt{1 - \lambda + \lambda x_n}. \tag{21}$$

Prove that his optimal choice is $\lambda = 1$, but that $U'(X_1^*)$ is not in general a (multiple of an) equivalent martingale measure:

$$\mathbb{E}[U'(X_1^*)(S_1 - S_0)] \neq 0. \tag{22}$$

Example 4. (Cuoco and Liu (2000)). This is an important example, generalising a number of other papers in the subject: Cvitanić and Karatzas (1992, 1993), El Karoui, Peng and Quenez (1997), Cuoco and Cvitanić (1998), for example. The wealth process X of the agent satisfies

$$dX_t = X_t \Big[r_t dt + \pi_t \cdot \big\{ \sigma_t dW_t + (b_t - r_t \mathbf{1}) dt \big\} + g(t, \pi_t) dt \Big] - c_t dt, \tag{23}$$

with $X_0 = x$, where W is an n-dimensional Brownian motion, b, r, $V \equiv \sigma\sigma^T$, V^{-1} are all bounded processes, and there is a uniform Lipschitz bound on g: for some $\gamma < \infty$,

$$|g(t, x, \omega) - g(t, y, \omega)| \leq \gamma |x - y|$$

for all x, y, t and ω. The only unconventional term in the dynamics (23) is the term involving g, about which we assume:

(i) for each $x \in I\!R^n$, $(t, \omega) \mapsto g(t, x, \omega)$ is an optional process;

(ii) for each $t \in [0, T]$ and $\omega \in \Omega$, $x \mapsto g(t, x, \omega)$ is concave and upper semi-continuous.

(iii) $g(t, 0, \omega) = 0$ for all $t \in [0, T]$ and $\omega \in \Omega$.

The agent has the objective of maximising

$$\mathbb{E}\Big[\int_0^T U(s, c_s) ds + U(T, X_T) \Big], \tag{24}$$

where we assume that for every $t \in [0, T]$ the map $c \mapsto U(t, c)$ is strictly increasing, strictly concave, and satisfies the Inada conditions.

EXERCISE 4. Show that the dual problem is to find

$$\inf_Y \mathbb{E}\Big[\int_0^T \tilde{U}(t, Y_t)\, dt + \tilde{U}(T, Y_T) + xY_0 \Big] \tag{25}$$

where the process Y solves

$$Y_t^{-1} dY_t = V_t^{-1} \big(r_t \mathbf{1} - b_t - \nu_t \big) \cdot \sigma_t dW_t - \big(r_t + \tilde{g}(t, \nu_t) \big) dt \tag{26}$$

for some adapted process ν bounded by γ, and where $\tilde{g}$ is the convex dual of g.

Example 5. (Cvitanic and Karatzas (1996)). This is a simple example incorporating transaction costs, where the holdings X_t of cash and the holdings Y_t of the sole share at time t obey

$$dX_t = r_t X_t dt + (1-\varepsilon)dM_t - (1+\delta)dL_t - c_t dt, \tag{27}$$
$$dY_t = Y_t(\sigma_t dW_t + \rho_t dt) - dM_t + dL_t, \tag{28}$$

where M and L are increasing processes, with the usual uniform boundedness assumptions on σ, ρ, σ^{-1} and r. The investor starts with initial holdings $(X_0, Y_0) = (x, y)$, and chooses the pair (L, M) and the consumption rate c so as to achieve his objective. Suppose that this is to

$$\sup \mathbb{E}\Big[\int_0^T U(c_s)ds + u(X_T, Y_T)\Big],$$

where we restrict to strategies lying always in the solvency region:

$$X_t + (1-\varepsilon)Y_t \geq 0, \quad X_t + (1-\delta)Y_t \geq 0 \quad \forall t$$

Exercise 5. By introducing Lagrangian semimartingales

$$d\xi_t = \xi_t(\alpha_t dW_t + \beta_t dt),$$
$$d\eta_t = \eta_t(a_t dW_t + b_t dt),$$

show that the dual form of the problem is

$$\inf \mathbb{E}\Big[\int_0^T \tilde{U}(t, \xi_t)dt + \tilde{u}(\xi_T, \eta_T) + x\xi_0 + y\eta_0\Big],$$

where the dual feasibility conditions to be satisfied are

$$\begin{aligned} \beta_t &= -r_t \\ b_t &= -\rho_t - \sigma_t a_t \\ 1 - \varepsilon \leq \tfrac{\eta_t}{\xi_t} &\leq 1 + \delta, \end{aligned}$$

and $\tilde{U}$, $\tilde{u}$ are the convex dual functions of U, u respectively.

Remark 2. The Lagrangian semimartingales of this example are related to the dual processes (Z^0, Z^1) of Cvitanic and Karatzas by

$$Z_t^0 = B_t\xi_t, \quad Z_t^1 = S_t\eta_t,$$

where B is the bond price process, and S is the stock price process solving

$$dB_t = B_t dt, \quad dS_t = S_t(\sigma_t dW_t + \rho_t dt).$$

Example 6. (Broadie, Cvitanic and Soner (1998)). This interesting example finds the minimum super-replicating price of a European-style contingent claim with a constraint on the portfolio process. Thus we are looking at the problem of Example 2, with the dynamics (23) of Example 4, specialized by assuming that σ, b and r are positive constants, and that g takes the form

$$g(t,x) = 0 \quad \text{if } x \in C; \quad = -\infty \quad \text{if } x \notin C,$$

where C is some closed convex set. We suppose that the contingent claim to be super-replicated, B, takes the form $B = \varphi(S)$ for some non-negative lower semi-continuous function φ, where S is the vector of share prices, solving

$$dS^i_t = S^i_t \Big[\sum_j \sigma_{ij} dW^j_t + \rho_i dt \Big].$$

The interest rate r, volatility matrix σ and drift ρ are all assumed constant, and σ is square and invertible.

EXERCISE 6. (Cvitanic and Karatzas (1993).) Show that the super-replication price for B is given by

$$\sup_\nu \mathbb{E}[Y_T(\nu)B],$$

where $Y(\nu)$ solves (26) with initial condition $Y_0 = 1$.

3 Dual Problems Made Difficult

Example 5 shows us that a formulation broad enough to embrace problems with transaction costs has to consider vector-valued asset processes; it is not sufficient to consider the aggregate wealth of the investor. This can be done, and *is* done in Klein and Rogers (2001), but in the present context we shall restrict ourselves to a univariate formulation of the problem. This saves us from a certain amount of careful convex analysis, which is not particularly difficult, and gives a result which will cover all the earlier examples apart from Example 5.

The main result below, Theorem 1, proves that under certain conditions, the value of the primal problem, expressed as a supremum over some set, is equal to the value of the dual problem, expressed as an infimum over some other set. It is important to emphasise that the Theorem does *not* say that the supremum in the primal problem is attained in the set, because such a result is not true in general without further conditions, and is typically very deep: see the paper of Kramkov and Schachermayer (1999), which proves that in the situation of Example 3 a further condition on the utility is needed in general to deduce that the value of the primal problem is attained. The result presented here is at its heart an application of the Minimax Theorem, and the argument is modelled on the argument of Kramkov and Schachermayer (1999).

To state the result, we set up some notation and introduce various conditions, a few of which (labelled in bold face) are typically the most difficult to check. Let $(S, \mathcal{S}, \mu)$

be some finite measure space, and let $L^0_+(S,\mathcal{S},\mu)$ denote the cone of non-negative functions in $L^0(S,\mathcal{S},\mu)$, a closed convex set usually abbreviated to L^0_+. We shall suppose that for each $x \geq 0$ we have a subset $\mathcal{X}(x)$ of L^0_+ with the properties

(X1) *$\mathcal{X}(x)$ is convex;*

(X2) *$\mathcal{X}(\lambda x) = \lambda\mathcal{X}(x)$ for all $\lambda > 0$;*

(X3) *if $g \in L^0_+$ and $g \leq f$ for some $f \in \mathcal{X}(x)$, then $g \in \mathcal{X}(x)$ also;*

(X4) *the constant function $\mathbf{1} : s \mapsto 1$ is in $\mathcal{X}$,*

where we have used the notation

$$\mathcal{X} \equiv \bigcup_{x\geq 0} \mathcal{X}(x) = \bigcup_{x\geq 0} x\mathcal{X}(1) \tag{29}$$

in stating (X4).

For the dual part of the story, we need for each $y \geq 0$ a subset $\mathcal{Y}(y) \subseteq L^0_+$ with the property

(Y1) *$\mathcal{Y}(y)$ is convex;*

(Y2) *for each $y \geq 0$, the set $\mathcal{Y}(y)$ is closed under convergence in μ-measure.*

We introduce the notation

$$\mathcal{Y} \equiv \bigcup_{y\geq 0} \mathcal{Y}(y) \tag{30}$$

for future use.

The primal and dual quantities are related by the key polarity property

(XY) *for all $f \in \mathcal{X}$ and $y \geq 0$*

$$\sup_{g\in\mathcal{Y}(y)} \int fg\, d\mu = \inf_{x\in\Psi(f)} xy$$

where we have used the notation

$$\Psi(f) = \{x \geq 0 : f \in \mathcal{X}(x)\}.$$

Properties (X2) and (X3) give us immediately that for any $f \in \mathcal{X}$ there is some $\xi(f) \geq 0$ such that

$$\Psi(f) = (\xi(f), \infty) \quad \text{or} \quad [\xi(f), \infty);$$

as yet, we do not know whether the lower bound is in $\Psi(f)$ or not, but we *can* say for $f \in \mathcal{X}(x)$ we must have $\xi(f) \leq x$. It also follows from (XY) that

$$\int fg\, d\mu \leq xy \qquad f \in \mathcal{X}(x), g \in \mathcal{Y}(y). \tag{31}$$

Using (X4), we see from (31) that in fact $\mathcal{Y} \subseteq L^1_+$.

IMPORTANT REMARK. We shall see in examples that often we take in (Y1) some convex set $\mathcal{Y}_0(y)$ of exponential semimartingales started from y, and it is in general not at all clear that the condition (Y2) will be satisfied for these. However, if we let $\mathcal{Y}(y)$ denote the closure in $L^0(\mu)$ of $\mathcal{Y}_0(y)$, this remains convex, now satisfies (Y2) by definition, and by Fatou's lemma

$$\sup_{g \in \mathcal{Y}(y)} \int fg d\mu = \sup_{g \in \mathcal{Y}_0(y)} \int fg d\mu,$$

so all we need to confirm (XY) is to check the statements for $g \in \mathcal{Y}_0(y)$. There is of course a price to pay, and that is that the statement of the main result is somewhat weaker.

Finally, we shall need a utility function $U : S \times \mathbb{R}^+ \to \mathbb{R} \cup \{-\infty\}$ with the basic properties

(U1) *$s \mapsto U(s,x)$ is $\mathcal{S}$-measurable for all $x \geq 0$;*

(U2) *$x \mapsto U(s,x)$ is concave, differentiable, strictly increasing, and finite-valued on $(0,\infty)$ for every $s \in \mathcal{S}$.*

We shall without comment assume that the definition of U has been extended to the whole of $S \times \mathbb{R}$ by setting $U(s,x) = -\infty$ if $x < 0$. Differentiability is not essential, but makes some subsequent statements easier.

We also impose the Inada-type conditions:

(U3) *if*

$$\varepsilon_n(s) \equiv U'(s,n), \tag{32}$$

we suppose that

$$\varepsilon_n(s) \to 0 \quad \mu - \text{a.e.} \tag{33}$$

as $n \to \infty$, and that there exists some n_0 such that

$$\int |\varepsilon_{n_0}(s)| \, \mu(ds) < \infty. \tag{34}$$

One consequence of this is that

$$U(s,x)/x \to 0 \quad (x \to \infty), \tag{35}$$

and another is that for any $z > 0$, the supremum defining the convex dual function $\tilde{U}$ is attained:

$$\begin{aligned} \tilde{U}(s,z) &\equiv \sup_{x>0} \{U(s,x) - xz\} \\ &= \max_{x>0} \{U(s,x) - xz\}. \end{aligned} \tag{36}$$

(U4) *the concave function*

$$\underline{u}(\lambda) \equiv \inf_{s \in \mathcal{S}} U(s,\lambda)$$

is finite-valued on $(0,\infty)$ *and satisfies the Inada condition*

$$\lim_{\lambda \downarrow 0} \frac{\partial \underline{u}}{\partial \lambda} = \infty;$$

Important remark[5] . We can in fact relax the condition (U4) to the simpler

(U4') *the concave function*

$$\underline{u}(\lambda) \equiv \inf_{s \in \mathcal{S}} U(s,\lambda)$$

is finite-valued on $(0,\infty)$.

The reason, explained in more detail in Section 6, is that we can always approximate a given utility uniformly to within any given $\varepsilon > 0$ by one satisfying the Inada condition at 0.

We impose one last (very slight) condition of a technical nature:

(U5) *there exists* $\psi \in \mathcal{X}$*, strictly positive, such that for all* $\varepsilon \in (0,1)$

$$U'(s, \varepsilon\psi(s)) \in L^1(S, \mathcal{S}, \mu);$$

Next we define the functions $u : I\!R^+ \to [-\infty,\infty)$ and $\tilde{u} : I\!R^+ \to (-\infty,\infty]$ by

$$u(x) \equiv \sup_{f \in \mathcal{X}(x)} \int U(s, f(s))\mu(ds) \tag{37}$$

and

$$\tilde{u}(y) \equiv \inf_{g \in \mathcal{Y}(y)} \int \tilde{U}(s, g(s))\mu(ds). \tag{38}$$

To avoid vacuous statements, we make the following finiteness assumption:

(F) for some $f_0 \in \mathcal{X}$ and $g_0 \in \mathcal{Y}$ we have

$$\int U(s, f_0(s))\mu(ds) > -\infty,$$
$$\int \tilde{U}(s, g_0(s))\mu(ds) < \infty.$$

Notice immediately one simple consequence of (F) and (31): if $f \in \mathcal{X}(x)$ and $g \in \mathcal{Y}(y)$,

$$\begin{aligned}\int U(s, f(s))\,\mu(ds) &\le \int \Big[\, U(s, f(s)) - f(s)g(s) \,\Big]\,\mu(ds) + xy \\ &\le \int \tilde{U}(s, g(s))\,\mu(ds) + xy. \end{aligned} \tag{39}$$

[5] Thanks to Nizar Touzi for noticing that the Inada condition at zero is unnecessary.

Taking $g = g_0$ in this inequality tells us that u is finite-valued, and taking $f = f_0$ tells us that $\tilde{u}$ is finite-valued.

Theorem 1. *Under the conditions stated above, the functions u and $\tilde{u}$ are dual:*

$$\tilde{u}(y) = \sup_{x \geq 0} \left[\, u(x) - xy \,\right], \tag{40}$$

$$u(x) = \inf_{y \geq 0} \left[\, \tilde{u}(y) + xy \,\right]. \tag{41}$$

Proof. Firstly, notice that part of what we have to prove is very easy: indeed, using the inequality (39), by taking the supremum over $f \in \mathcal{X}(x)$ and the infimum over $g \in \mathcal{Y}(y)$ we have that

$$\tilde{u}(y) \geq u(x) - xy \tag{42}$$

for any non-negative x and y. The other inequality is considerably more difficult, and is an application of the Minimax Theorem.

Define the function $\Phi : \mathcal{X} \times \mathcal{Y} \to [-\infty, \infty)$ by

$$\Phi(f, g) \equiv \int \left[\, U(s, f(s)) - f(s)g(s) \,\right] \mu(ds), \tag{43}$$

and introduce the sets

$$\mathcal{B}_n \equiv \{f \in L^\infty_+(S, \mathcal{S}, \mu) : 0 \leq f(s) \leq n \; \forall s\}. \tag{44}$$

Then $\mathcal{B}_n$ is convex, and compact in the topology $\sigma(L^\infty, L^1)$. We need the following result.

Lemma 1. *For each $y \geq 0$, for each $g \in \mathcal{Y}(y)$, the map $f \mapsto \Phi(f, g)$ is upper semicontinuous on $\mathcal{B}_n$ and is sup-compact: for all a*

$$\{f \in \mathcal{B}_n : \Phi(f, g) \geq a\} \quad \text{is } \sigma(L^\infty, L^1)\text{-compact.}$$

PROOF. The map $f \mapsto \int fg d\mu$ is plainly continuous in $\sigma(L^\infty, L^1)$ on $\mathcal{B}_n$, so it is sufficient to prove the upper semicontinuity assertion in the case $g = 0$,

$$f \mapsto \int U(s, f(s)) \; \mu(ds).$$

Once we have upper semicontinuity, the compactness statement is obvious. So the task is to prove that for any $a \in I\!\!R$, the set

$$\{f \in \mathcal{B}_n : \int U(s, f(s)) \; \mu(ds) \geq a\}$$
$$= \bigcap_{\varepsilon > 0} \{f \in \mathcal{B}_n : \int U(s, f(s) + \varepsilon\psi(s)) \; \mu(ds) \geq a\}$$

is $\sigma(L^\infty, L^1)$-closed. The equality of these two sets is immediate from the Monotone Convergence Theorem and the fact that $\psi \in \mathcal{X}$, and the fact that $U(s,\cdot)$ is increasing for all s. We shall prove that for each $\varepsilon > 0$ the set

$$N_\varepsilon = \{f \in \mathcal{B}_n : \int U(s, f(s) + \varepsilon\psi(s))\ \mu(ds) < a\}$$

is open in $\sigma(L^\infty, L^1)$. Indeed, if $h \in \mathcal{B}_n$ is such that

$$\int U(s, h(s) + \varepsilon\psi(s))\ \mu(ds) = a - \delta < a,$$

we have by (U2) that for any $f \in \mathcal{B}_n$

$$\begin{aligned}
&\int U(s, f(s) + \varepsilon\psi(s))\ \mu(ds) \\
&\le \int \Big[U(s, h(s) + \varepsilon\psi(s)) + (f(s) - h(s))U'(s, h(s) + \varepsilon\psi(s)) \Big]\ \mu(ds) \\
&\le a - \delta + \int (f(s) - h(s))U'(s, h(s) + \varepsilon\psi(s))\ \mu(ds)
\end{aligned}$$

Since $U'(s, h(s) + \varepsilon\psi(s)) \in L^1(S, \mathcal{S}, \mu)$ by (U5), this exhibits a $\sigma(L^\infty, L^1)$-open neighbourhood of h which is contained in N_ε, as required. □

We now need the Minimax Theorem, Theorem 7 on p 319 of Aubin and Ekeland (1984), which we state here for completeness, expressed in notation adapted to the current context.

Minimax Theorem. *Let B and Y be convex subsets of vector spaces, B being equipped with a topology. If*

(MM1) for all $g \in Y$, $f \mapsto \Phi(f,g)$ is concave and upper semicontinuous;

(MM2) for some $g_0 \in Y$, $f \mapsto \Phi(f, g_0)$ is sup-compact;

(MM3) for all $f \in B$, $g \mapsto \Phi(f,g)$ is convex,

then

$$\sup_{f\in B} \inf_{g \in Y} \Phi(f,g) = \inf_{g\in Y} \sup_{f \in B} \Phi(f,g),$$

and the supremum on the left-hand side is attained at some $\bar{f} \in B$.

We therefore have

$$\sup_{f\in\mathcal{B}_n} \inf_{g\in\mathcal{Y}(y)} \Phi(f,g) = \inf_{g\in\mathcal{Y}(y)} \sup_{f\in\mathcal{B}_n} \Phi(f,g). \tag{45}$$

From this,

$$\sup_{f\in\mathcal{B}_n} \inf_{g\in\mathcal{Y}(y)} \Phi(f,g) = \inf_{g\in\mathcal{Y}(y)} \int \tilde{U}_n(s, g(s))\ \mu(ds) \equiv \tilde{u}_n(y), \tag{46}$$

say, where

$$\tilde{U}_n(s,z) \equiv \sup\{U(s,x) - zx : 0 \le x \le n\} \uparrow \tilde{U}(s,z). \tag{47}$$

Consequently, $\tilde{u}_n(y) \le \tilde{u}(y)$.

Using the property (XY) going from the second to the third line, we estimate

$$\begin{aligned}
\tilde{u}_n(y) = \sup_{f\in\mathcal{B}_n} \inf_{g\in\mathcal{Y}(y)} \Phi(f,g) &= \sup_{f\in\mathcal{B}_n} \inf_{g\in\mathcal{Y}(y)} \int \{U(s,f(s)) - f(s)g(s)\}\mu(ds) \\
&= \sup_{f\in\mathcal{B}_n} \Big[\int U(s,f(s))\mu(ds) - \sup_{g\in\mathcal{Y}(y)} \int fg\, d\mu\Big] \\
&= \sup_{f\in\mathcal{B}_n} \Big[\int U(s,f(s))\mu(ds) - \inf_{x\in\Psi(f)} xy\Big] \\
&= \sup_{f\in\mathcal{B}_n} \sup_{x\in\Psi(f)} \Big[\int U(s,f(s))\mu(ds) - xy\Big] \\
&\le \sup_{f\in\mathcal{X}} \sup_{x\in\Psi(f)} \Big[\int U(s,f(s))\mu(ds) - xy\Big] \\
&= \sup_{x\in C} \sup_{f\in\mathcal{X}(x)} \Big[\int U(s,f(s))\mu(ds) - xy\Big] \\
&= \sup_{x\in C} \Big[u(x) - xy\Big]
\end{aligned}$$

The $\tilde{u}_n(y)$ clearly increase with n, so the proof will be complete provided we can prove that

$$\lim_n \tilde{u}_n(y) = \tilde{u}(y), \tag{48}$$

Suppose that $g_n \in \mathcal{Y}(y)$ are such that

$$\tilde{u}_n(y) \le \int \tilde{U}_n(s, g_n(s))\mu(ds) \le \tilde{u}_n(y) + n^{-1}. \tag{49}$$

Using Lemma A1.1 of Delbaen and Schachermayer (1994), we can find a sequence

$$h_n \in \mathrm{conv}(g_n, g_{n+1}, \ldots)$$

in $\mathcal{Y}(y)$ which converge μ-almost everywhere to a function h taking values in $[0,\infty]$; because of (Y2), $h \in \mathcal{Y}(y)$. Moreover, because of (X4) and (31), the limit must be almost everywhere finite, and hence

$$\lim_n \int \tilde{U}_n(s, h_n(s))\mu(ds) = \lim_n \tilde{u}_n(y).$$

From the definition (32) of $\varepsilon_n(s)$, it is immediate that

$$\tilde{U}_n(s,z) = \tilde{U}(s,z) \quad \text{if } y \ge \varepsilon_n(s).$$

One last fact is needed, which we prove later.

Proposition 1. *The family* $\{\tilde{U}(s, \varepsilon_n(s) + g(s))^- : g \in \mathcal{Y}(y), n \geq n_0\}$ *is uniformly integrable.*

Using these facts, we have the inequalities

$$\begin{aligned}
\tilde{u}(y) &\leq \int \tilde{U}(s, h(s))\, \mu(ds) \\
&\leq \liminf_n \int \tilde{U}(s, \varepsilon_n(s) + h(s))\, \mu(ds) \\
&\leq \liminf_n \liminf_{m \geq n} \int \tilde{U}(s, \varepsilon_n(s) + h_m(s))\, \mu(ds) \\
&\leq \liminf_n \liminf_{m \geq n} \int \tilde{U}_m(s, \varepsilon_n(s) + h_m(s))\, \mu(ds) \\
&\leq \liminf_n \liminf_{m \geq n} \int \tilde{U}_m(s, h_m(s))\, \mu(ds) \\
&\leq \tilde{u}(y),
\end{aligned}$$

as required. □

Proof of Proposition 1. The argument here is a slight modification of that of Kramkov and Schachermayer (1999); we include it for completeness. Firstly, we note that

$$\begin{aligned}
-\tilde{U}(s, z) &\equiv \inf_{x \geq 0} \{ xz - U(s, x) \} \\
&\leq \inf_{x \geq 0} \{ xz - \underline{u}(x) \} \\
&\equiv \psi(z),
\end{aligned}$$

say. We suppose that $\sup_a \psi(a) = \infty$, otherwise there is nothing to prove, and let $\varphi : (\psi(0), \infty) \to (0, \infty)$ denote its convex increasing inverse. We have

$$\lim_{x \to \infty} \frac{\varphi(x)}{x} = \lim_{y \to \infty} \frac{y}{\psi(y)} = \lim_{t \downarrow 0} \frac{\underline{u}'(t)}{t\underline{u}'(t) - \underline{u}(t)} = \lim_{t \downarrow 0} \frac{\int_t^1 \underline{u}''(ds)}{\int_t^1 s\underline{u}''(ds)} = \infty,$$

using the property (U4). Now we estimate

$$\begin{aligned}
\int \varphi(\tilde{U}(s, \varepsilon_n(s) + g(s))^-)\mu(ds) &\leq \int \varphi(\max\{0, \psi(\varepsilon_n(s) + g(s))\}\mu(ds) \\
&\leq \varphi(0)\mu(S) + \int \varphi(\psi(\varepsilon_n(s) + g(s)))\mu(ds) \\
&= \varphi(0)\mu(S) + \int \{\varepsilon_n(s) + g(s)\}\mu(ds).
\end{aligned}$$

This is bounded uniformly in $g \in \mathcal{Y}(y)$, by (X4), (31) and (U3). □

There is a useful little corollary of this proposition.

Corollary 1. *For each $y \geq 0$, there is some $g \in \mathcal{Y}(y)$ for which the infimum defining $\tilde{u}(y)$ in (38) is attained.*

Differentiability of U implies strict convexity of $\tilde{U}$, which in turn implies uniqueness of the minimising g.

Proof. Take $g_n \in \mathcal{Y}(y)$ such that

$$\tilde{u}(y) \leq \int \tilde{U}(s, g_n(s))\mu(ds) \leq \tilde{u}(y) + n^{-1}. \tag{50}$$

By again using Lemma A1.1 of Delbaen and Schachermayer (1994) we may suppose that the g_n are μ-almost everywhere convergent to limit g, still satisfying the inequalities (50). Now by Proposition 1 and Fatou's lemma,

$$\tilde{u}(y) \leq \int \tilde{U}(s, g(s))\mu(ds) \leq \liminf_n \int \tilde{U}(s, g_n(s))\mu(ds) \leq \tilde{u}(y),$$

as required. The uniqueness assertion is immediate. □

4 Dual Problems Made Honest

So far, I have made a number of soft comments about how to identify the dual problem in a typical situation, and have stated and proved a general abstract result which I claim turns the heuristic recipe into a proved result. All we have to do in any given example is simply to verify the conditions of the previous Section, under which Theorem 1 was proved; in this Section, we will see this verification done in full for the Cuoco-Liu example, and you will be able to judge for yourself just how 'simply' this verification can be done in practice!

To recall the setting, the dynamics of the agent's wealth is given by (23):

$$dX_t = X_t \Big[r_t dt + \pi_t \cdot \big\{ \sigma_t dW_t + (b_t - r_t \mathbf{1})dt \big\} + g(t, \pi_t)dt \Big] - c_t dt, \tag{51}$$

with $X_0 = x$ and where the processes $b, r, V \equiv \sigma\sigma^T, V^{-1}$ are all bounded processes, and there is a uniform Lipschitz bound on g: for some $\gamma < \infty$,

$$|g(t, x, \omega) - g(t, y, \omega)| \leq \gamma |x - y|$$

for all x, y, t and ω. The function g is assumed to be concave and vanishing at zero in its second argument, and the agent aims to maximise his objective (24):

$$\mathbb{E}\Big[\int_0^T U(s, c_s)ds + U(T, X_T) \Big], \tag{52}$$

where for every $t \in [0, T]$ the map $c \mapsto U(t, c)$ is strictly increasing, strictly concave, and satisfies the Inada conditions.

We believe that the dual form of the problem is given by the solution to Exercise 4, namely, to find

$$\inf_Y \mathbb{E}\Big[\int_0^T \tilde{U}(t, Y_t)\,dt + \tilde{U}(T, Y_T) + xY_0\Big] \tag{53}$$

where the process Y solves

$$Y_t^{-1} dY_t = V_t^{-1}\big(r_t\mathbf{1} - b_t - \nu_t\big) \cdot \sigma_t dW_t - \big(r_t + \tilde{g}(t, \nu_t)\big)dt \tag{54}$$

for some adapted process ν bounded by γ, and where $\tilde{g}$ is the convex dual of g.

Now we have to cast the problem into the form of Section 3 so that we may apply the main result, Theorem 1. For the finite measure space $(S, \mathcal{S}, \mu)$ we take

$$S = [0, T] \times \Omega, \quad \mathcal{S} = \mathcal{O}[0, T], \quad \mu = (\text{Leb}[0, T] + \delta_T) \times \mathbb{P},$$

where $\mathcal{O}[0, T]$ is the optional[6] σ-field restricted to $[0, T]$. The set $\mathcal{X}(x)$ is the collection of all *bounded* optional $f : S \mapsto I\!\!R^+$ such that for some non-negative (X, c) satisfying (51), for all ω,

$$f(t, \omega) \le c(t, \omega), \qquad (0 \le t < T), \qquad f(T, \omega) \le X(T, \omega). \tag{55}$$

Remark 3. The assumption that f is bounded is a technical detail without which it appears very hard to prove anything. The conclusion is not in any way weakened by this assumption, though, as we shall discuss at the end.

Next we define $\mathcal{Y}_1(y)$ to be the set of all solutions to (26) with initial condition $Y_0 = y$. From this we define the set $\mathcal{Y}_0(y)$ to be the collection of all non-negative adapted processes h such that for some $Y \in \mathcal{Y}_1(y)$

$$h(t, \omega) \le Y(t, \omega) \quad \mu\text{-almost everywhere.}$$

Finally, we define a utility function $\varphi : S \times I\!\!R^+ \mapsto I\!\!R \cup \{-\infty\}$ in the obvious way:

$$\varphi((t, \omega), x) = U(t, x),$$

and we shall slightly abuse notation and write U in place of φ henceforth.

We have now defined the objects in terms of which Theorem 1 is stated, and we have to prove that they have the required properties.

(X1) If (X^1, c^1) and (X^2, c^2) solve (51) with portfolio processes π^1 and π^2, say, taking any $\theta_1 = 1 - \theta_2 \in [0, 1]$ and defining

[6] That is, the σ-field generated by the stochastic intervals $[\tau, \infty)$ for all stopping times τ of the Brownian motion. See, for example, Section VI.4 of Rogers and Williams (2000).

$$\begin{aligned}\bar{X} &= \theta^1 X^1 + \theta^2 X^2,\\ \bar{c} &= \theta^1 c^1 + \theta^2 c^2,\\ \bar{\pi} &= \frac{\theta^1 \pi^1 X^1 + \theta^2 \pi^2 X^2}{\bar{X}}\end{aligned}$$

we find immediately that

$$d\bar{X}_t = \bar{X}_t \Big[\, r_t dt + \bar{\pi}_t \cdot \big\{ \sigma_t dW_t + (b_t - r_t \mathbf{1})dt \big\} + g(t, \bar{\pi}_t)dt - \Psi_t dt \, \Big] - \bar{c}_t dt,$$

where

$$\Psi_t = g(t, \bar{\pi}_t) - \Big[\theta^1 X^1_t g(t, \pi^1_t) + \theta^2 X^2_t g(t, \pi^2_t) \, \Big] / \bar{X} \geq 0,$$

using the concavity of g. It easy to deduce from this that

$$X^*_t - \bar{X}_t \geq 0, \qquad 0 \leq t \leq T,$$

where X^* is the solution to

$$dX^*_t = X^*_t \Big[\, r_t dt + \bar{\pi}_t \cdot \big\{ \sigma_t dW_t + (b_t - r_t \mathbf{1})dt \big\} + g(t, \bar{\pi}_t)dt \, \Big] - \bar{c}_t dt,$$

starting at x. Hence $(X^*, \bar{c})$ solves (51) with portfolio $\bar{\pi}$, and the convex combination $(\bar{X}, \bar{c})$ is in $\mathcal{X}(x)$. Hence $\mathcal{X}(x)$ is convex.

(X2) and (X3) are trivial.

(X4) By taking $\pi = 0$, and using the fact that r is bounded, we see from the dynamics (23) that for some small enough $\varepsilon > 0$ we can achieve a constant consumption stream $c_t = \varepsilon$ with terminal wealth $X_T \geq \varepsilon$. This establishes (X4).

(Y1) The proof of convexity of $\mathcal{Y}(y)$ is analogous to the proof of property (X1).

(Y2) Because of the global Lipschitz assumption on g, it can be shown that in fact $\mathcal{Y}_0(y)$ is closed in $L^0(\mu)$ for all $y \geq 0$; see Klein and Rogers (2001) for details. Hereafter, $\mathcal{Y}(y)$ will be defined to be the closure in $L^0(\mu)$ of $\mathcal{Y}_0(y)$; they *are* equal, but no use will be made of this fact.

The properties of the utility, and the finiteness assumption are as quickly dealt with: properties (U1) and (U2) are evident, while the remaining properties must be checked on each particular case. For example, if the utility has separable form

$$U(s, c) = h(s) f(c)$$

then provided h is bounded, and f is strictly increasing, concave and satisfies the Inada conditions, the conditions (U3)–(U5) are satisfied. For the finiteness condition (F), we must once again check this for each particular case.

(XY) So far so good; now the work begins. The heart of the proof lies in establishing the duality relation (XY) but in reality only one half of (XY) presents any challenges.

If (c, X) solves (51) and if Y solves (53), then using Itô's formula gives us

$$\begin{aligned} d(X_tY_t) &= X_tY_t\big\{\pi_t\cdot\sigma_t dW_t + (r_t\mathbf{1} - b_t - \nu_t)\cdot\sigma_t^{-1}dW_t \\ &\qquad\qquad +(-\pi_t\cdot\nu_t + g(t,\pi_t) - \tilde{g}(t,\nu_t))dt\big\} - c_tY_tdt \\ &\doteq X_tY_t[(g(t,\pi_t) - \pi_t\cdot\nu_t - \tilde{g}(t,\nu_t)]dt - c_tY_tdt, \end{aligned} \tag{56}$$

where the symbol $\doteq$ signifies that the two sides differ by a local martingale. From this, using the definition of $\tilde{g}$, we conclude that

$$X_tY_t + \int_0^t Y_s c_s\, ds \quad \text{is a non-negative supermartingale,}$$

which leads immediately to the inequality

$$X_0Y_0 \geq \mathbb{E}[\int_0^T Y_s c_s\, ds + X_TY_T].$$

Thus we have half of (XY); if $f \in \mathcal{X}(x)$, then

$$\sup_{h\in\mathcal{Y}(y)} \int fh\, d\mu \leq xy,$$

and so

$$\sup_{h\in\mathcal{Y}(y)} \int fh\, d\mu \leq \inf_{x\in\Psi(f)} xy.$$

Remark 4. If we took *any* f dominated as at (55), but not necessarily bounded, the above analysis still holds good, and $\int fh\, d\mu \leq xy$ for all $h \in \mathcal{Y}(y)$.

What remains now is to prove that if $f \in \mathcal{X}$ and

$$\sup_{h\in\mathcal{Y}(1)} \int fh\, d\mu \equiv \xi \leq x \tag{57}$$

then $f \in \mathcal{X}(x)$, for which it is evidently equivalent to prove that $f \in \mathcal{X}(\xi)$ in view of (X2). Notice the interpretation of what we are required to do here. It could be that the given $f \in \mathcal{X}$ were dominated as at (55) by some (c, X) which came from a very large initial wealth x_0, but that the value ξ were much smaller than x_0. What we now have to do is to show that the consumption plan and terminal wealth defined by f can actually be financed by the smaller initial wealth ξ.

The argument requires three steps:

Step 1: Show that the supremum in (57) is attained at some $Y^* \in \mathcal{Y}_1(1)$:

$$\xi = \mathbb{E}\left[\int_0^T Y_s^* f_s\, ds + Y_T^* f_T\right]; \tag{58}$$

Step 2: Use the Y^* from Step 1 to construct a (conventional) market in which the desired consumption stream and terminal wealth f can be achieved by replication, using investment process π^* with initial wealth ξ;

Step 3: By considering the process ν related to π^* by duality, show that in fact the investment process π^* replicates f in the original market.

Here is how the three steps of the argument are carried out.

Step 1. We prove the following.

Proposition 2. *There exists $Y^* \in \mathcal{Y}(1)$, with corresponding process ν^* in (54), such that*

$$\xi = \mathbb{E}[\int_0^T Y_s^* f_s \, ds + f_T Y_T^*]. \tag{59}$$

PROOF. There exist processes $\nu^{(n)}$ bounded by the constant γ such that if $Y^{(n)}$ is the solution to (54) using $\nu = \nu^{(n)}$ with initial value 1, then

$$\mathbb{E}[\int_0^T Y_s^{(n)} f_s \, ds + f_T Y_T^{(n)}] > \xi - 2^{-n}. \tag{60}$$

We need to deduce convergence of the $\nu^{(n)}$ and the $Y^{(n)}$. Using once again Lemma A1.1 of Delbaen and Schachermayer (1994), we have a sequence

$$\bar{\nu}^{(n)} \in \text{conv}(\nu^{(n)}, \nu^{(n+1)}, \ldots)$$

which converges μ-a.e. to a limit ν^*, since all the $\nu^{(n)}$ are bounded. Now because of the boundedness assumptions, and because $\tilde{g} \geq 0$, it is obvious from (54) that the sequence $Y^{(n)}$ is bounded in L^2, and so is uniformly integrable. Using the fact that f was assumed bounded, we deduce the convergence in (60). $\square$

Step 2. Write the dual process Y^* in product form as

$$Y_t^* \equiv Z_t^* \beta_t^* \equiv \exp(\int_0^t \psi_s \cdot dW_s - \frac{1}{2}\int_0^t |\psi_s|^2 \, ds). \exp(-\int_0^t r_s^* \, ds)$$

where

$$\begin{aligned} \psi_t &= \sigma_t^{-1}(r_t \mathbf{1} - b_t - \nu_t^*), \\ r_t^* &= r_t + \tilde{g}(t, \nu_t^*). \end{aligned}$$

By the Cameron-Martin-Girsanov theorem (see, for example, Rogers and Williams (2000), IV.38), the martingale Z^* defines a new measure $\mathbb{P}^*$ via the recipe $d\mathbb{P}^* = Z_T^* \, d\mathbb{P}$, and in terms of this

$$dW_t = dW_t^* + \psi_t dt$$

where W^* is a $\mathbb{P}^*$-Brownian motion. The term β^* is interpreted as a stochastic discount factor, and the equality (59) can be equivalently expressed as

$$\xi = \mathbb{E}^*[\int_0^T \beta_s^* f_s \, ds + \beta_T^* f_T].$$

The bounded $\mathbb{P}^*$-martingale

$$\begin{aligned} M_t &\equiv \mathbb{E}^*\left[\int_0^T \beta_s^* f_s\, ds + \beta_T^* f_T \middle| \mathcal{F}_t\right] \\ &= \int_0^t \beta_s^* f_s\, ds + \mathbb{E}^*\left[\int_t^T \beta_s^* f_s\, ds + \beta_T^* f_T \middle| \mathcal{F}_t\right] \end{aligned}$$

has an integral representation

$$M_t = \xi + \int_0^t \theta_s \cdot dW_s^*$$

for some previsible square-integrable integrand θ (see, for example, Rogers and Williams (2000), IV.36). Routine calculations establish that the process

$$X_t^* \equiv (M_t - \int_0^t \beta_s^* f_s\, ds)/\beta_t^*$$

satisfies

$$\begin{aligned} dX_t^* &= X_t^*\left[r_t^* dt + \pi_t^* \cdot \sigma_t\, dW_t^*\right] - f_t dt \\ &= X_t^*\left[r_t^* dt + \pi_t^* \cdot \sigma_t\, dW_t + \pi_t^* \cdot (b_t - r_t \mathbf{1} + \nu_t^*) dt\right] - f_t dt \\ &= X_t^*\left[r_t dt + \pi_t^* \cdot \left\{\sigma_t\, dW_t + (b_t - r_t \mathbf{1}) dt\right\} + g(t, \pi_t^*) dt + \varepsilon_t dt\right] - f_t dt \\ &\equiv X_t^*\left[dZ_t/Z_t + \varepsilon_t dt\right] - f_t dt, \end{aligned} \tag{61}$$

where we have used the notations

$$\begin{aligned} \pi_t^* &\equiv (\sigma_t^{-1})^T \cdot \theta_t/(\beta_t^* X_t^*), \\ \varepsilon_t &\equiv \tilde{g}(t, \nu_t^*) - g(t, \pi_t^*) + \pi_t^* \cdot \nu_t^* \geq 0, \\ dZ_t &= Z_t\left[r_t dt + \pi_t^* \cdot \left\{\sigma_t\, dW_t + (b_t - r_t \mathbf{1}) dt\right\} + g(t, \pi_t^*) dt\right]. \end{aligned}$$

We have moreover that $X_0^* = \xi$ and $X_T^* = f_T$, so *provided we could show that ε were zero,* we have constructed a solution pair (c, X) to (51) for which $c_s = f_s$ for $0 \leq s < T$, and $X_T = f_T$; *we therefore have the required conclusion $f \in \mathcal{X}(\xi)$ provided $\varepsilon = 0$.*

Step 3. The goal is now clear, and the proof is quite straightforward. If we construct the process X as solution to

$$dX_t = X_t(dZ_t/Z_t) - f_t dt, \qquad X_0 = \xi,$$

then we have from (61) that

$$d(X_t^* - X_t) = (X_t^* - X_t) dZ_t/Z_t + \varepsilon_t X_t^* dt$$

and hence

$$X_t^* - X_t = Z_t \int_0^t \left(\varepsilon_s X_s^*/Z_s\right)\, ds \geq 0. \tag{62}$$

In particular, since X^* is a bounded process, $-X$ is bounded below by some constant. Now take[7] a process ν such that for all t

$$\tilde{g}(t, \nu_t) = g(t, \pi_t^*) - \pi_t^* \cdot \nu_t,$$

and form the process $Y \in \mathcal{Y}(1)$ from ν according to (54); because of the boundedness of r, b, and ν, Y is dominated by an integrable random variable. Repeating the calculation of (56) gives us

$$d(X_t Y_t) \doteq -Y_t f_t dt;$$

thus $-X_t Y_t - \int_0^t Y_s f_s\ ds$ is a local martingale bounded below by an integrable random variable, therefore a supermartingale. Hence

$$\begin{aligned}
\xi &\leq \mathbb{E}\left[\int_0^T Y_s f_s\ ds + Y_T X_T\right] \\
&\leq \mathbb{E}\left[\int_0^T Y_s f_s\ ds + Y_T X_T^*\right] && (63) \\
&= \mathbb{E}\left[\int_0^T Y_s f_s\ ds + Y_T f_T\right] \\
&\leq \xi && (64)
\end{aligned}$$

where inequality (63) will be strict unless $\int_0^T \varepsilon_s\ ds = 0$ almost surely, from (62), and (64) is just the definition of ξ. The conclusion that

$$\tilde{g}(t, \nu_t^*) = g(t, \pi_t^*) - \pi_t^* \cdot \nu_t^*$$

μ-a.e. now follows, and so $f \in \mathcal{X}(\xi)$, as required. □

Remark 5. We have assumed that $\mathcal{X}(x)$ consists of bounded processes, and have used this boundedness hypothesis in several places. Some such boundedness restriction does appear to be needed in general; however, we expect to argue at the end that no real loss of generality has occurred. For example, in this situation if we were to have taken the larger feasible set $\bar{\mathcal{X}}(x)$ to be the set of *all* optional processes f dominated by some (c, X) solving (51), not just the bounded ones, then the new value

$$\bar{u}(x) \equiv \sup_{f \in \bar{\mathcal{X}}(x)} \int U(s, f(s))\mu(ds)$$

[7] If $\tilde{g}$ is strictly convex, there is no problem, as the value of $\hat{\nu}$ is unique. More generally we need a measurable selection, but we omit further discussion of this point. In any case, the function g can be uniformly approximated above and below by smooth concave functions, and the result will hold for these; see Section 6 for further discussion.

certainly is no smaller than the value $u(x)$ we have been working with. But as we remarked earlier, for any $f \in \bar{\mathcal{X}}(x)$ and $h \in \mathcal{Y}(y)$,

$$\int fh \, d\mu \leq xy,$$

and the inequality analogous to (42) holds:

$$\tilde{u}(y) \geq \bar{u}(x) - xy,$$

since the argument leading to (39) works just as well for $f \in \bar{\mathcal{X}}(x)$. We therefore have

$$\tilde{u}(y) \geq \bar{u}(x) - xy \geq u(x) - xy;$$

taking the supremum over x, the two ends of these inequalities have been proved to give the same value, so the result holds good for $\bar{u}$ as well.

5 Dual Problems Made Useful

If duality can only turn one impossible problem into another, then it is of no practical value. However, as I shall show in this Section by presenting in full the analysis of the problem of Broadie, Cvitanic and Soner (1998), there really *are* situations where duality can do more than 'reveal the structure' of the original problem, and can in fact lead to a complete solution.

For this problem, introduced in Section 2, it turns out to be more convenient to work with log prices. Since the original share prices satisfy

$$dS_t^i = S_t^i \Big[\sum_j \sigma_{ij} dW_t^j + \rho_i dt \Big],$$

the log prices $X_t^i \equiv \log S_t^i$ satisfy

$$dX_t^i = \sigma_{ij} dW_t^j + b_i dt, \tag{65}$$

where $b_i \equiv \rho_i - a_{ii}/2$, $a \equiv \sigma\sigma^T$, and we use the summation convention in (65). We further define

$$\psi(X) \equiv \log \varphi(e^X),$$

so that the aim is to super-replicate the random variable $B = \exp(\psi(X_T))$. According to the result of Exercise 6, we must compute

$$\sup_\nu \mathbb{E}[Y_T(\nu)B], \tag{66}$$

where $Y(\nu)$ solves

$$Y_t^{-1} dY_t = \sigma^{-1}\big(r\mathbf{1} - \rho - \nu_t\big) \cdot dW_t - \big(r + \tilde{g}(\nu_t)\big)dt$$

with initial condition $Y_0 = 1$. But the dual form (66) of the problem can be tackled by conventional HJB techniques. Indeed, if we define

$$f(t, X) \equiv \sup_{\nu} \mathbb{E}\left[\frac{Y_T(\nu)}{Y_t(\nu)} \, B \;\middle|\; X_t = X \right],$$

then for any process ν we shall have

$$z_t \equiv Y_t(\nu) f(t, X_t) \quad \text{is a supermartingale, and a martingale for optimal } \nu. \tag{67}$$

Abbreviating $Y(\nu)$ to Y, the Itô expansion of z gives us

$$dz_t \doteq Y_t \big[\, \mathcal{L} f(t, X_t) - (r + \tilde{g}(\nu_t)) f(t, X_t) + \nabla f(t, X_t) \cdot (r\mathbf{1} - \rho - \nu_t) \,\big] dt,$$

where $\mathcal{L}$ is the generator of X,

$$\mathcal{L} \equiv \frac{1}{2} a_{ij} D_i D_j + b_i D_i + \frac{\partial}{\partial t},$$

and $D_i \equiv \partial / \partial x_i$. The drift term in z must be non-positive in view of the optimality principle (67), so we conclude that

$$\begin{aligned}
0 &= \sup_{\nu} \big[\, \mathcal{L} f(t, X_t) - (r + \tilde{g}(\nu)) f(t, X_t) + \nabla f(t, X_t) \cdot (r\mathbf{1} - \rho - \nu) \,\big] \\
&= \mathcal{L} f(t, X_t) + \nabla f(t, X_t) \cdot (r\mathbf{1} - \rho) - r f(t, X_t) \\
&\qquad\qquad - f(t, X_t) \inf_{\nu} \big\{ \tilde{g}(\nu) + \nabla (\log f)(t, X_t) \cdot \nu \big\} \\
&= \mathcal{L} f(t, X_t) + \nabla f(t, X_t) \cdot (r\mathbf{1} - \rho) - r f(t, X_t) - f(t, X_t) g(\nabla (\log f)(t, X_t)).
\end{aligned}$$

For this to be possible, it has to be that

$$\nabla (\log f)(t, x) \in C \quad \text{for all } (t, x), \tag{68}$$

and the equation satisfied by f will be

$$0 = \mathcal{L}_0 f(t, X) - r f(t, X),$$

where $\mathcal{L}_0 = \mathcal{L} + (r - a_{ii}/2) D_i$ is the generator of X in the risk-neutral probability. Thus f satisfies the pricing equation, so we have[8]

$$f(t, x) = \mathbb{E}^* \big[\exp(\hat{\psi}(X_T)) \,\big|\, X_t = x \big] \tag{69}$$

for some function $\hat{\psi}$ such that

$$f(T, x) = \exp(\hat{\psi}(x)).$$

In order that we have super-replicated, we need to have $\hat{\psi} \geq \psi$, and in order that the gradient condition (68) holds we must also have

[8] ... using $\mathbb{P}^*$ to denote the risk-neutral pricing measure ...

$$\nabla\hat{\psi}(x) \in C \quad \forall x. \tag{70}$$

What is the smallest function $\hat{\psi}$ which satisfies these two conditions? If a function ψ_0 satifies the gradient condition (70) and is at least as big as ψ everywhere, then for any x and x' the Mean Value Theorem implies that there is some $x'' \in (x, x')$ such that

$$\psi_0(x) - \psi_0(x') = (x - x') \cdot \nabla\psi_0(x'')$$

so

$$\begin{aligned}\psi_0(x) &\geq \psi(x') + \inf_{v\in C}(x - x') \cdot v \\ &= \psi(x') - \delta(x - x'),\end{aligned}$$

where $\delta(v) \equiv \sup\{-x\cdot v : x \in C\}$ is the support function of C. Taking the supremum over x', we learn that

$$\psi_0(x) \geq \Psi(x) \equiv \sup_y\{\psi(x - y) - \delta(y)\}.$$

Now clearly $\Psi \geq \psi$ (think what happens when $y = 0$), but we have further that Ψ satisfies the gradient condition (70). Indeed, if not, there would be x and x' such that

$$\begin{aligned}\Psi(x') &> \Psi(x) + \sup_{v\in C}(x' - x) \cdot v \\ &= \Psi(x) + \delta(x - x').\end{aligned}$$

However, using the convexity and positive homogeneity of δ, we have

$$\begin{aligned}\Psi(x') &\equiv \sup_y\{\psi(x' - y) - \delta(y)\} \\ &= \sup_z\{\psi(x - z) - \delta(x' - x + z)\} \\ &= \sup_z\{\psi(x - z) - \delta(z) + \delta(z) - \delta(x' - x + z)\} \\ &\leq \Psi(x) + \delta(x - x'),\end{aligned}$$

a contradiction.

This establishes the gradient condition (68) for $t = T$, but why should it hold for other t as well? The answer lies in the expression (69) for the solution, together with the fact that X is a drifting Brownian motion, because for any h we have

$$\begin{aligned}&\frac{\Big(f(t, x + h) - f(t, x)\Big)}{|h| f(t, x)} \\ &= \frac{\mathbb{E}^*\big[\, |h|^{-1}(\exp(\hat{\psi}(X_T + x + h)) - \exp(\hat{\psi}(X_T + x))) \,\big|\, X_t = 0\big]}{\mathbb{E}^*\big[\exp(\hat{\psi}(X_T + x))) \,\big|\, X_t = 0\big]} \\ &\to \frac{\mathbb{E}^*\big[\, \nabla\hat{\psi}(X_T + x)\exp(\hat{\psi}(X_T + x))) \,\big|\, X_t = 0\big]}{\mathbb{E}^*\big[\exp(\hat{\psi}(X_T + x))) \,\big|\, X_t = 0\big]}\end{aligned}$$

as $|h| \to 0$ under suitable conditions. The final expression is clearly in C, since it is the expectation of a random vector which always takes values in the convex set C.

Remark 6. For another example of an explicitly-soluble dual problem, see the paper of Schmock, Shreve and Wystup (2001).

6 Taking Stock

We have seen how the Lagrangian/Hamiltonian/Pontryagin approach to a range of constrained optimisation problems can be carried out very simply, and can be very effective. The recipe in summary is to introduce a Lagrange multiplier process, integrate by parts, and look at the resulting Lagrangian; the story of Section 1 and the examples of Section 2 are so simple that one could present them to a class of MBA students. However, mathematicians ought to grapple with the next two Sections as well, to be convinced that the approach *can* be turned into proof.

What remains? There are many topics barely touched on in these notes, which could readily be expanded to twice the length if due account were to be taken of major contributions so far ignored. Let it suffice to gather here a few remarks under disparate headings, and then we will be done.

1. Links with the work of Bismut. Bismut's (1975) paper and its companions represent a remarkable contribution, whose import seems to have been poorly digested, even after all these years. The original papers were presented in a style which was more scholarly than accessible, but what he did in those early papers *amounts to the same as we have done here.* To amplify that claim, let me take a simple case of his analysis as presented in the 1975 paper, using similar notation, and follow it through according to the recipe of this account. Bismut takes a controlled diffusion process[9]

$$dx = \sigma(t, x, u)dW + f(t, x, u)dt$$

with the objective of maximising

$$\mathbb{E} \int_0^T L(t, x, u)\, dt.$$

The coefficients σ, f and L may be suitably stochastic, x is n-dimensional, W is d-dimensional, u is q-dimensional. By the method advanced here, we would now introduce a n-dimensional Lagrange multiplier process[10]

$$dp = bdt + HdW$$

and absorb the dynamics into the Lagrangian by integrating-by-parts; we quickly obtain the Lagrangian[11]

[9] For economy of notation, any superfluous subscript t is omitted from the symbol for a process.

[10] Bismut's notation. H is $n \times d$.

[11] For matrices A and B of the same dimension, $\langle A, B\rangle$ is the L^2-inner product $tr(AB^T)$.

$$\begin{aligned}\Lambda = \mathbb{E}\bigg[\int_0^T L(t,x,u)\,dt - [p\cdot x]_0^T + \int_0^T (x\cdot b + \langle H, \sigma(t,x,u)\rangle)\,dt \\ + \int_0^T p\cdot f(t,x,u)\,dt\bigg] \\ = \mathbb{E}\bigg[p_0\cdot x_0 - p_T\cdot x_T + \int_0^T (x\cdot b + \mathcal{H}(t,x,u))\,dt\bigg],\end{aligned}$$

where the Hamiltonian $\mathcal{H}$ of Bismut's account is defined by

$$\mathcal{H}(t,x,u;p,H) \equiv L(t,x,u) + p\cdot f(t,x,u) + \langle H, \sigma(t,x,u)\rangle).$$

Various assumptions will be needed for suprema to be finite and uniquely attained, but our next step would be to maximise the Lagrangian Λ over choice of u, which would lead us to solve the equations

$$\frac{\partial \mathcal{H}}{\partial u} = 0,$$

and maximising Λ over x would lead to the equations

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial x} &= -b, \\ p_T &= 0.\end{aligned}$$

These are the equations which Bismut obtains. The final section of Bismut (1973) explains the relation between the solution obtained by the Lagrangian approach, and the (value-function) solution of the classical dynamic programming approach.

2. Does this dual approach work for all problems? The answer is 'Yes' and 'No', just as it is for the standard dynamic programming approach. We have seen a number of problems where the dual problem can be expressed quite simply, and generally this is not hard to do, but moving from there to an explicit solution can only rarely be achieved[12] (indeed, *just* as in the standard dynamic programming approach, where it is a few lines' work to find the HJB equation to solve, but only rarely can an explicit solution be found.)

During the workshop, Nizar Touzi showed me uncomfortably many examples for which the dual approach described here offered no useful progress; as he stressed, problems where the control affects the volatility of the processes are usually difficult. Here then is an interesting and very concrete example which seems to be hard to deal with.

Example. An investor has wealth process X satisfying

$$dX_t = \theta_t dS_t, \quad X_0 = x,$$

where θ_t is the number of shares held at time t, and S_t is the share price process satisfying

[12] But Chow (1997) proposes methods for approximate numerical solution if all else fails.

$$dS_t/S_t = \frac{v}{N-\theta_t}\, dW_t, \quad S_0 = 1,$$

where $v > 0$ and $N > 0$. The modelling idea is that there are only N shares in total, and that as the number of shares held by the rest of the market falls, the volatility of the price increases. The agent's objective is to maximise

$$\mathbb{E}\big[\, U(X_T - S_T) \,\big].$$

Introducing Lagrangian semimartingales $d\xi = \xi(adW + bdt)$ for the dynamics of X and $d\eta = \eta(\alpha dW + \beta dt)$ for the dynamics of S, we form the Lagrangian in the usual fashion:

$$\begin{aligned}
\Lambda = \sup \mathbb{E}\Bigg[& U(X_T - S_T) - ([X\xi]_0^T - \int_0^T X b\xi \, dt - \int_0^T a\xi \frac{\theta v S}{N-\theta}\, dt) \\
& + [S\eta]_0^T - \int_0^T S\eta\beta \, dt - \int_0^T \frac{\alpha \eta v S}{N-\theta}\, dt \Bigg] \\
= \sup \mathbb{E}\Bigg[& \tilde{U}(\xi_T) + X_0\xi_0 - \eta_0 S_0 + \int_0^T \frac{S}{N-\theta}\, (a\xi\theta v - \eta\beta(N-\theta) - \alpha v \eta)\, dt \Bigg] \qquad (71) \\
= \mathbb{E}\Bigg[& \tilde{U}(\xi_T) + X_0\xi_0 - \eta_0 S_0 \Bigg]
\end{aligned}$$

provided that the dual-feasibility conditions

$$\begin{aligned}
\xi_T &= \eta_T \\
b &= 0 \\
\alpha\eta &\geq Na \\
v\alpha + N\beta &\geq 0
\end{aligned}$$

are satisfied. The last two come from inspecting the integral in (71); if the bracket was positive for any value of θ in $(0, N)$, then by taking S arbitrarily large we would have an unbounded supremum for the Lagrangian. The form of this dual problem looks quite intractable; the multiplier processes are constrained to be equal at time T, but the bounds on the coefficients of ξ and η look tough. Any ideas?

3. Links to the work of Kramkov-Schachermayer. Anyone familiar with the paper of Kramkov and Schachermayer (1999) (hereafter, KS) will see that many of the ideas and methods of this paper owe much to that. The fact that this paper has not so far mentioned the asymptotic elasticity property which was so important in KS is because what we have been concerned with here is *solely* the equality of the values of the primal and dual problem; the asymptotic elasticity condition of KS was used at the point where they showed that the supremum in the primal problem was attained, and this is not something that we have cared about. The paper KS works in a general semimartingale context, where the duality result (XY) are really very deep; on the other hand, the problem considered in KS is to optimise the expected utility of terminal wealth, so the problem is the simplest one in terms of objective. It is undoubtedly an important goal to generalise the study of optimal investment and consumption problems to the semimartingale setting; it was after all only when stochastic calculus was extended from the Brownian to the general semimartingale framework that we came to understand the rôle of key objects (semimartingales among them!). Such an extension remains largely unfinished at the time of writing.

4. Equilbria. From an economic point of view, the study of the optimal behaviour of a single agent is really only a step on the road towards understanding an equilibrium of many agents interacting through a market. In the simplest situation of a frictionless market without portfolio constraints, the study of the equilibrium is quite advanced; see Chapter 4 of Karatzas and Shreve (1998). However, once we incorporate portfolio constraints of the type considered for example by Cuoco and Liu (2000), it become very difficult to characterise the equilibria of the system. There are already some interesting studies (see Section 4.8 of Karatzas and Shreve (1998) for a list), but it is clear that much remains to be done in this area.

5. Smoothing utilities. In a remark after stating property (U4), I said that the Inada condition at zero was not really needed; the reason is the following little lemma, which shows that we may always uniformly approximate any given utility by one which does satisfy the Inada condition at 0.

Lemma 2. *Let C be a closed convex cone with non-empty interior, and suppose that $U : C \to I\!R^d \cup \{-\infty\}$ is concave, finite-valued on* $\text{int}(C)$*, and increasing in the partial order of C. Assume that the dual cone C^* also has non-empty interior. Then for any $\varepsilon > 0$ we can find $U_\varepsilon, U^\varepsilon : C \to I\!R^d \cup \{-\infty\}$ such that*

$$U(x) - \varepsilon \le U_\varepsilon(x) \le U(x) \le U^\varepsilon(x) \le U(x) + \varepsilon$$

for all $x \in$ $\text{int}(C)$, and such that U_ε, U^ε, are strictly concave, strictly increasing, differentiable, and satisfy the Inada condition for any $x \in$ $\text{int}(C)$

$$\lim_{\lambda \downarrow 0} \frac{\partial}{\partial \lambda} U_\varepsilon(\lambda x) = +\infty = \lim_{\lambda \downarrow 0} \frac{\partial}{\partial \lambda} U^\varepsilon(\lambda x).$$

Proof. Suppose that $\{x_1, \ldots, x_d\} \subseteq C$ is a basis, and that $\{y_1, \ldots, y_d\} \subseteq C^*$ is a basis. Now the functions

$$\tilde{u}_+(y) \equiv \tilde{U}(y) + \frac{\varepsilon}{2} \exp\Big(- \sum_{j=1}^d \sqrt{x_j \cdot y} \Big),$$

$$\tilde{u}_-(y) \equiv \tilde{U}(y) + \frac{\varepsilon}{2} \exp\Big(- \sum_{j=1}^d \sqrt{x_j \cdot y} \Big) - \frac{\varepsilon}{2},$$

which sandwich $\tilde{U}$ to within $\varepsilon/2$, are strictly decreasing in y, and are strictly convex. The dual functions, $u_\pm$, are therefore differentiable in $\text{int}(C)$, increasing, and sandwich U to within $\varepsilon/2$. They may fail to be strictly concave, but by considering instead

$$u_{++}(x) \equiv u_+(x) + \frac{\varepsilon}{2}\{1 - \exp\Big(-\sum_{j=1}^{d}\sqrt{x_j \cdot y})\Big)\}$$

$$u_{--}(x) \equiv u_-(x) - \frac{\varepsilon}{2}\exp\Big(-\sum_{j=1}^{d}\sqrt{x_j \cdot y}\Big)$$

we even have the strict concavity as well, and the Inada condition is evident. □

Remark 7. The assumption of non-empty interior for C is not needed; if the interior is empty, we simply drop down to the subspace spanned by C, in which C has non-empty relative interior, and apply the Lemma there. If C contained a linear subspace, then because U is increasing in the order of C, it must be constant in the direction of that subspace, and we can drop down to the quotient space (which now contains no linear subspace) and work there instead.

7 Solutions to Exercises

Solution to Exercise 1. Expressing the expectation of $\int_0^T Y_s dX_s$ in the two ways, we get (assuming that the means of stochastic integrals dW are all zero)

$$\mathbb{E}\Big[X_T Y_T - X_0 Y_0 - \int_0^T Y_s\{\alpha_s X_s + \theta_s \sigma_s \beta_s\}ds\Big], \tag{72}$$

and

$$\mathbb{E}\Big[\int_0^T Y_s\{r_s X_s + \theta_s(\mu_s - r_s\mathbf{1}) - c_s\}ds\Big]. \tag{73}$$

The Lagrangian form now is

$$\begin{aligned}
\Lambda(Y) \equiv \sup_{X,c\geq 0,\theta} \mathbb{E}\Big[\int_0^T U(s,c_s)ds \\
&+ U(T,X_T) + \int_0^t Y_s\{r_s X_s + \theta_s(\mu_s - r_s\mathbf{1}) - c_s\}ds \\
&- X_T Y_T + X_0 Y_0 + \int_0^T Y_s\{\alpha_s X_s + \theta_s\sigma_s\beta_s\}ds\Big] \\
= \sup_{X,c\geq 0,\theta} \mathbb{E}\Big[\int_0^T \{U(s,c_s) - Y_s c_s\}ds + U(T,X_T) - X_T Y_T + X_0 Y_0 \\
&+ \int_0^T Y_s\{r_s X_s + \theta_s(\mu_s - r_s\mathbf{1}) + \alpha_s X_s + \theta_s\sigma_s\beta_s\}ds\Big].
\end{aligned} \tag{74}$$

Now the maximisation of (74) over $c \geq 0$ and $X_T \geq 0$ is very easy; we obtain

$$\Lambda(Y) = \sup_{X\geq 0,\theta} \mathbb{E}\Big[\int_0^T \tilde{U}(s,Y_s)ds + \tilde{U}(T,Y_T) + X_0Y_0 + \int_0^T Y_s\{r_sX_s + \theta_s(\mu_s - r_s\mathbf{1}) + \alpha_sX_s + \theta_s\sigma_s\beta_s\}ds\Big],$$

where $\tilde{U}(s,y) \equiv \sup_x[U(s,x) - xy]$ is the convex dual of U. The maximisation over $X_s \geq 0$ results in a finite value if and only if the complementary slackness condition

$$r_s + \alpha_s \leq 0 \qquad (75)$$

holds, and maximisation over θ_s results in a finite value if and only if the complementary slackness condition

$$\sigma_s\beta_s + \mu_s - r_s\mathbf{1} = 0 \qquad (76)$$

holds. The maximised value is then

$$\Lambda(Y) = \mathbb{E}\Big[\int_0^T \tilde{U}(s,Y_s)ds + \tilde{U}(T,Y_T) + X_0Y_0\Big]. \qquad (77)$$

The dual problem is therefore the minimisation of (77) with the complementary slackness conditions (75), (76). But in fact, since the dual functions $\tilde{U}(t,\cdot)$ are decreasing, a little thought shows that we want Y to be big, so that the 'discount rate' α will be as large as it can be, that is, the inequality (75) will actually hold with equality. This gives the stated form of the dual problem.

Solution to Exercise 2. We use the result of Example 0. Since the dual function $\tilde{u}_0$ of u_0 is just $-u_0$, we have from the dual problem to Example 0, (12), that

$$\begin{aligned}\sup \mathbb{E}[\, u_0(X_T - B)\,] &= \inf_Y \mathbb{E}[\, \tilde{u}_0(Y_T) - Y_TB + xY_0\,] \\ &= \inf_{Y_T\geq 0} \mathbb{E}[\, xY_0 - Y_TB\,].\end{aligned}$$

Clearly, this will be $-\infty$ if

$$x < \sup_{Y_T\geq 0} \mathbb{E}[\, Y_TB/Y_0\,],$$

and zero else. The statement (18) follows.

Solution to Exercise 3a. Introducing the positive Lagrangian semimartingale Y in exponential form

$$dY_t = Y_{t-}dz_t = Y_{t-}(dm_t + dA_t),$$

where m is a local martingale and A is a process of finite variation, and integrating by parts, we find that

$$\int_0^T Y_{t-} dX_t = X_T Y_T - X_0 Y_0 - \int_0^T X_{t-} dY_t - [X,Y]_T$$
$$= \int_0^T Y_{t-} H_t dS_t.$$

Hence the Lagrangian is

$$\begin{aligned}
\Lambda(Y) &\equiv \sup \mathbb{E}\Big[U(X_T) + \int_0^T Y_{t-} H_t dS_t - X_T Y_T + X_0 Y_0 \\
&\qquad + \int_0^T X_{t-} dY_t + [X,Y]_T \Big] \\
&= \sup \mathbb{E}\Big[U(X_T) - X_T Y_T + X_0 Y_0 + \int_0^T Y_{t-} H_t \, dS_t \\
&\qquad + \int_0^T X_{t-} Y_{t-} (dm_t + dA_t) + \int_0^T H_t Y_{t-} \, d[m,S]_t \Big] \\
&= \sup \mathbb{E}\Big[\tilde{U}(Y_T) + X_0 Y_0 + \int_0^T Y_{t-} \big(H_t \, (dS_t + d[m,S]_t) + X_{t-} dA_t \big) \Big]
\end{aligned}$$

if means of stochastic integrals with respect to local martingales are all zero. Maximising the Lagrangian over $X \geq 0$, we obtain the dual-feasibility condition that $dA \leq 0$. Next, by maximising over H we see that we must have $dS + d[m,S] \doteq 0$, from which

$$\begin{aligned}
d(XY) &= X_- dY + Y_- dX + d[X,Y] \\
&\doteq X_- Y_- dA + Y_- H(dS + d[m,S]) \\
&\doteq X_- Y_- dA
\end{aligned}$$

so that XY is a non-negative supermartingale.

Solution to Exercise 3b. Differentiating (21) with respect to λ, we find that condition (20) is exactly the condition for the derivative to be non-negative throughout $[-1,1]$. Hence the agent's optimal policy is just to invest all his money in the share. Could $U'(S_1)$ be an equivalent martingale measure? This would require

$$\mathbb{E}[U'(S_1)(S_1 - S_0)] = 0,$$

or equivalently,

$$\mathbb{E}[\sqrt{S_1}] = \mathbb{E}[1/\sqrt{S_1}].$$

It is clear that by altering the $(p_n)_{n \geq 0}$ slightly if necesary, this equality can be broken.

Solution to Exercise 4. Introduce the Lagrangian semimartingale Y satisfying

$$dY_t = Y_t \{ \, \alpha_t \cdot \sigma_t dW_t + \beta_t dt \, \} \tag{78}$$

and now develop the two different expressions for $\int Y dX$, firstly as

$$\begin{aligned}\int_0^T Y_t dX_t &= Y_T X_T - Y_0 X_0 - \int_0^T X_t \, dY_t - [X,Y]_T \\ &= Y_T X_T - Y_0 X_0 - \int_0^T X_t Y_t \big\{ \alpha_t \cdot \sigma_t dW_t + \beta_t dt \big\} - [X,Y]_T \\ &= Y_T X_T - Y_0 X_0 - \int_0^T X_t Y_t \big\{ \alpha_t \cdot \sigma_t dW_t + \beta_t dt + \alpha_t \cdot V_t \, \pi_t \, dt \big\} \\ &\doteq Y_T X_T - Y_0 X_0 - \int_0^T X_t Y_t \big\{ \beta_t + \alpha_t \cdot V_t \, \pi_t \big\} \, dt. \end{aligned} \tag{79}$$

The symbol $\doteq$ signifies that the two sides of the equation differ by a local martingale vanishing at zero. Next, we express $\int Y dX$ as

$$\begin{aligned}\int_0^T Y_t dX_t = \int_0^T Y_t X_t \Big[\, r_t dt + \pi_t \cdot \big\{ \sigma_t dW_t + (b_t - r_t \mathbf{1}) dt \big\} + g(t, \pi_t) dt \, \Big] \\ - \int_0^T Y_t c_t dt \\ \doteq \int_0^T Y_t X_t \Big[\, r_t + \pi_t \cdot (b_t - r_t \mathbf{1}) + g(t, \pi_t) \Big] dt - \int_0^T Y_t c_t dt. \end{aligned} \tag{80}$$

The Lagrangian form is now

$$\begin{aligned} \Lambda \equiv \mathbb{E} \bigg[\int_0^T U(s, c_s) ds + U(T, X_T) \\ + \int_0^T Y_t X_t \Big[\, r_t + \pi_t \cdot (b_t - r_t \mathbf{1}) + g(t, \pi_t) \Big] dt - \int_0^T Y_t c_t dt \\ - Y_T X_T + Y_0 X_0 + \int_0^T X_t Y_t \big\{ \beta_t + \alpha_t \cdot V_t \pi_t \big\} \, dt \bigg]. \end{aligned} \tag{81}$$

Maximising this over X_T and c gives

$$\begin{aligned} \Lambda = \mathbb{E} \bigg[\int_0^T \tilde{U}(t, Y_t) \, dt + \tilde{U}(T, Y_T) + Y_0 X_0 \\ + \int_0^T X_t Y_t \big\{ \beta_t + \alpha_t \cdot V_t \pi_t + r_t + \pi_t \cdot (b_t - r_t \mathbf{1}) + g(t, \pi_t) \big\} \, dt. \end{aligned} \tag{82}$$

We find on the way the dual feasibility conditions on Y that, almost surely, $Y_t \geq 0$ for almost every t, and and $Y_T \geq 0$, with strict inequality for t for which U_t is unbounded above.

Since we have that Y and X are both non-negative processes, maximising (82) over π amounts to maximising the expression

$$g(t, \pi) - \pi \cdot (r_t \mathbf{1} - b_t - V_t \alpha_t)$$

for each t; the maximised value of this expression can be written in terms of the convex dual $\tilde{g}(t,\cdot)$ of g as

$$\tilde{g}(t,\nu_t),$$

where ν is related to α by

$$\nu_t \equiv r_t\mathbf{1} - b_t - V_t\alpha_t. \tag{83}$$

Alternatively, we can express α in terms of ν as

$$\alpha_t = V_t^{-1}\big(r_t\mathbf{1} - b_t - \nu_t\big). \tag{84}$$

The value of (82) when so maximised over π is therefore

$$\begin{aligned} \Lambda = \mathbb{E}\bigg[\int_0^T & \tilde{U}(t,Y_t)\,dt + \tilde{U}(T,Y_T) + Y_0X_0 \\ & + \int_0^T X_tY_t\big\{\,\beta_t + r_t + \tilde{g}(t,\nu_t)\,\big\}\,dt\bigg]. \end{aligned} \tag{85}$$

Finally, we consider the maximisation of Λ over X. This leads to the dual-feasiblity condition

$$\beta_t + r_t + \tilde{g}(t,\nu_t) \le 0 \tag{86}$$

and a maximised value of the Lagrangian of the simple form

$$\Lambda = \mathbb{E}\bigg[\int_0^T \tilde{U}(t,Y_t)\,dt + \tilde{U}(T,Y_T) + Y_0X_0 \bigg].$$

But $X_0 = x$, so we now believe that the dual problem must be to find

$$\inf_Y \mathbb{E}\bigg[\int_0^T V(t,Y_t)\,dt + V(T,Y_T) + xY_0 \bigg]$$

where

$$Y_t^{-1}dY_t = V_t^{-1}\big(r_t\mathbf{1} - b_t - \nu_t\big)\cdot\sigma_t dW_t - \big(r_t + \tilde{g}(t,\nu_t)\big)dt - \varepsilon_t dt,$$

where ε is some non-negative process, from (86). Since we are looking to minimise the Lagrangian over Y, and since V_t is decreasing, it is clear that we should take $\varepsilon \equiv 0$, leading to the dynamics

$$Y_t^{-1}dY_t = V_t^{-1}\big(r_t\mathbf{1} - b_t - \nu_t\big)\cdot\sigma_t dW_t - \big(r_t + \tilde{g}(t,\nu_t)\big)dt$$

for Y.

SOLUTION TO EXERCISE 5. According to the machine, the Lagrangian is

$$\begin{aligned}\Lambda = \sup \mathbb{E}\Bigg[\int_0^T U(c_t)\,dt + u(X_T,Y_T) + \int_0^T (r_tX_t - c_t)\xi_t\,dt - X_T\xi_T + X_0\xi_0 \\ + \int_0^T X_t\xi_t\beta_t\,dt + \int \rho_tY_t\eta_t\,dt - Y_T\eta_T + Y_0\eta_0 + \int_0^T Y_t\eta_t(b_t+\sigma_ta_t)\,dt \\ + \int_0^T ((1-\varepsilon)\xi_t - \eta_t)\,dM_t + \int_0^T (\eta_t - (1+\delta)\xi_t)\,dL_t\Bigg]\end{aligned}$$

Maximising over increasing M and L, we see that we must have the dual feasibility conditions

$$(1-\varepsilon)\xi_t \le \eta_t \le (1+\delta)\xi_t$$

and the maximised value of the integrals dM and dL will be zero. The maximisation over c and over (X_T, Y_T) is straightforward and transforms the Lagrangian to

$$\begin{aligned}\Lambda = \sup \mathbb{E}\Bigg[\int_0^T \tilde{U}(\xi_t)\,dt + \tilde{u}(\xi_T,\eta_T) + X_0\xi_0 + Y_0\eta_0 \\ + \int_0^T X_T\xi_T(r_t+\beta_t)\,dt + \int_0^T Y_t\eta_t(\rho_t + b_t + \sigma_ta_t)\Bigg].\end{aligned}$$

Maximising over X and Y yields the dual feasibility conditions

$$r_t + \beta_t \le 0 \tag{87}$$

$$\rho_t + b_t + \sigma_ta_t \le 0 \tag{88}$$

with the final form of the Lagrangian as

$$\mathbb{E}\Bigg[\int_0^T \tilde{U}(\xi_t)\,dt + \tilde{u}(\xi_T,\eta_T) + X_0\xi_0 + Y_0\eta_0\Bigg].$$

The by now familiar monotonicity argument shows that in trying to minimise this over multipliers (ξ,η) we would have the two dual-feasibility conditions (87) and (88) satisfied with equality.

Solution to Exercise 6. At least at the heuristic level, which is all we are concerned with for the moment, this exercise follows from Example 4 in the same way that Exercise 2 was derived from Example 0. Just follow through the solution to Exercise 4 assuming that $U(t,.) = 0$ for $t < T$, and $U(T, X_T) = u_0(X_T - \varphi(S_T))$.

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The Problem of Super-replication under Constraints

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Summary. These notes present an overview of the problem of super-replication under portfolio constraints. We start by examining the duality approach and its limitations. We then concentrate on the direct approach in the Markov case which allows to handle general large investor problems and gamma constraints. In the context of the Black and Scholes model, the main result from the practical view-point is the so-called *face-lifting* phenomenon of the payoff function.

Key words: Super-replication, duality, dynamic programming, viscosity solutions, Hamilton-Jacobi-Bellman equation.
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1 Introduction

There is a large literature on the problem of super-replication in finance, i.e. the minimal initial capital which allows to hedge some given contingent claim at some terminal time T. Put in a stochastic control terms, the value function of the super-replication problem is the minimal initial data of some controlled process (the wealth process) which allows to hit some given target at time T. This stochastic control problem does not fit in the class of *standard* problems as presented in the usual textbooks, see e.g. [15]. This may explain the important attraction that this problem had on mathematicians.

In its simplest form, this problem is an alternative formulation of the Black and Scholes theory in terms of a stochastic control problem. The Black and Scholes solution appears naturally as a (degenerate) dual formulation of the super-replication problem. However, real financial markets are subject to constraints. The most popular example is the case of incomplete markets which was studied by Harrisson and Kreps (1979), and developed further by ElKaroui and Quenez (1995) in the diffusion case. The effect of the no short-selling constraint has been studied by Jouini and Kallal

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(1995). The dual formulation in the general convex constraints framework has been obtained by Cvitanić and Karatzas (1993) in the diffusion case and further extended by Föllmer and Kramkov (1997) to the general semimartingale case.

In the general constrained portfolio case, the above-mentioned dual formulation does not close the problem: except the complete market case, it provides an alternative stochastic control problem. The good news is that this problem is formulated in standard form. But there is still some specific complications since the controls are valued in unbounded sets. We are then in the context of singular control problems which typically exhibit a *jump* in the terminal condition. The main point is to characterize precisely this *face-lifting* phenomenon.

However, the duality approach has not been successful to solve more general super-replication problems. Namely, the dual formulation of the general large investor problem is still open. The same comment prevails for the super-replication problem under *gamma* constraints, i.e. constraints on the unbounded variation part of the portfolio. We provide a treatment of these problems which avoids the passage from the dual formulation. The key-point is an original dynamic programming principle stated directly on the initial formulation of the super-replication problem.

Further implications of this new dynamic programming principle in the field of differential geometry are reported in [23], [24] and [25]. The main result of these papers is to prove a stochastic representation for geometric equations in terms of a stochastic target problem.

2 Problem Formulation

2.1 The Financial Market

Given a finite time horizon $T > 0$, we shall consider throughout these notes a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with a standard Brownian motion $B = \{(B_1(t), \ldots, B_d(t)), 0 \le t \le T\}$ valued in $\mathbb{R}^d$, and generating the ($\mathbb{P}-$augmentation of the) filtration $\mathbb{F}$. We denote by ℓ the Lebesgue measure on $[0, T]$.

The financial market consists of a non-risky asset S^0 normalized to unity, i.e. $S^0 \equiv 1$, and d risky assets with price process $S = (S^1, \ldots, S^d)$ whose dynamics is defined by a stochastic differential equation. More specifically, given a vector process μ valued in $\mathbb{R}^n$, and a matrix-valued process σ valued in $\mathbb{R}^{n\times n}$, the price process S^i is defined as the unique strong solution of the stochastic differential equation:

$$S^i(0) \;=\; s^i \;,\;\; dS^i(t) \;=\; S^i(t)\left[b^i(t)dt + \sum_{j=1}^{d} \sigma^{ij}(t)dB^j(t)\right] \;; \qquad (1)$$

here b and σ are assumed to be bounded $\mathbb{F}-$adapted processes.

Remark 1. The normalization of the non-risky asset to unity is, as usual, obtained by discounting, i.e. taking the non-risky asset as a *numéraire*.

In the financial literature, σ is known as the *volatility* process. We assume it to be invertible so that the *risk premium* process

$$\lambda_0(t) := \sigma(t)^{-1}b(t) , \quad 0 \le t \le T ,$$

is well-defined. Throughout these notes, we shall make use of the process

$$Z_0(t) = \mathcal{E}\left(-\int_0^t \lambda_0(r)'dB(r)\right) := \exp\left(-\int_0^t \lambda_0(r)'dB(r) - \frac{1}{2}\int_0^t |\lambda_0(r)|^2\right) ,$$

where prime denotes transposition.

Standing Assumption. The volatility process σ satisfies:

$$\mathbb{E}\left[\exp\frac{1}{2}\int_0^T |\sigma'\sigma|^{-1}\right] < \infty \text{ and } \sup_{[0,T]} |\sigma'\sigma|^{-1} < \infty \ \mathbb{P}-\text{a.s.}$$

Since b is bounded, this condition ensures that the process λ_0 satisfies the Novikov condition $\mathbb{E}[\exp\int_0^T |\lambda_0|^2/2] < \infty$, and we have $\mathbb{E}[Z_0(T)] = 1$. The process Z_0 is then a martingale, and induces the probability measure $\mathbb{P}_0$ defined by:

$$\mathbb{P}_0(A) := \mathbb{E}[Z_0(t)\mathbf{1}_A] \text{ for all } A \in \mathcal{F}(t) , \ 0 \le t \le T .$$

Clearly $\mathbb{P}_0$ is equivalent to the original probability measure $\mathbb{P}$. By Girsanov's Theorem, the process

$$B_0(t) := B(t) + \int_0^t \lambda_0(t)dt , \quad 0 \le t \le T ,$$

is a standard Brownian motion under $\mathbb{P}_0$.

2.2 Portfolio and Wealth Process

Let $W(t)$ denote the wealth at time t of some investor on the financial market. We assume that the investor allocates continuously his wealth between the non-risky asset and the risky assets. We shall denote by $\pi^i(t)$ the proportion of wealth invested in the $i-th$ risky asset. This means that

$$\pi^i(t)W(t) \text{ is the amount invested at time } t \text{ in the } i-th \text{ risky asset.}$$

The remaining proportion of wealth $1-\sum_{i=1}^d \pi^i(t)$ is invested in the non-risky asset.

The *self-financing condition* states that the variation of the wealth process is only affected by the variation of the price process. Under this condition, the wealth process satisfies:

$$dW(t) = W(t)\sum_{i=1}^{d}\pi^i(t)\frac{dS^i(t)}{S^i(t)}$$
$$= W(t)\pi(t)'[b(t)dt + \sigma(t)dB(t)] \;=\; W(t)\pi(t)'\sigma(t)dB_0(t)\;. \qquad (2)$$

Hence, the investment strategy π should be restricted so that the above stochastic differential equation has a well-defined solution. Also $\pi(t)$ should be based on the information available at time t. This motivates the following definition.

Definition 1. *An investment strategy is an $\mathbb{F}-$adapted process π valued in $\mathbb{R}^d$ and satisfying $\int_0^T |\sigma'\pi|^2(t)dt < \infty$ $\mathbb{P}-$a.s.*
We shall denote by $\mathcal{A}$ the set of all investment strategies.

Clearly, given an initial capital $w \geq 0$ together with an investment strategy π, the stochastic differential equation (2) has a unique solution

$$W_w^\pi(t) := w\mathcal{E}\left(\int_0^t \pi(r)'\sigma(r)dB_0(r)\right)\;,\quad 0 \leq t \leq T\;.$$

We then have the following trivial, but very important, observation:

$$W_w^\pi \text{ is a } \mathbb{P}_0\text{−supermartingale}\;, \qquad (3)$$

as a non-negative local martingale under $\mathbb{P}_0$.

2.3 Problem Formulation

Let K be a closed convex subset of $\mathbb{R}^d$ containing the origin, and define the set of constrained strategies:

$$\mathcal{A}_K := \{\pi \in \mathcal{A} \;:\; \pi \in K\; \ell \otimes \mathbb{P} - \text{a.s.}\}\;.$$

The set K represents some constraints on the investment strategies.

Example 1. Incomplete market: taking $K = \{x \in \mathbb{R}^d \;:\; x^i = 0\}$, for some integer $1 \leq i \leq d$, means that trading on the $i-$th risky asset is forbidden.

Example 2. No short-selling constraint: taking $K = \{x \in \mathbb{R}^d \;:\; x^i \geq 0\}$, for some integer $1 \leq i \leq d$, means that the financial market does not allow to sell short the $i-$th asset.

Example 3. No borrowing constraint: taking $K = \{x \in \mathbb{R}^d \;:\; x^1 + \ldots + x^d \leq 1\}$ means that the financial market does not allow to sell short the non-risky asset or, in other word, borrowing from the bank is not available.

Now, let G be a non-negative $\mathcal{F}(T)-$ measurable random variable. The chief goal of these notes is to study the following stochastic control problem

$$V(0) := \inf\{w \in \mathbb{R} \;:\; W_w^\pi(T) \;\geq\; G\,\mathbb{P} - \text{a.s. for some } \pi \in \mathcal{A}_K\}\;. \qquad (4)$$

The random variable G is called a *contingent claim* in the financial mathematics literature, or a *derivative asset* in the financial engineering world. Loosely speaking, this is a contract between two counterparts stipulating that the seller has to pay G at time T to the buyer. Therefore, $V(0)$ is the minimal initial capital which allows the seller to face without risk the payment G at time T, by means of some clever investment strategy on the financial market.

We conclude this section by summarizing the main results which will be presented in these notes.

1. We start by proving that existence holds for the problem $V(0)$ under very mild conditions, i.e. there exists a constrained investment strategy $\pi \in \mathcal{A}_K$ such that $W^{\pi}_{V(0)}(T) \geq G\ \mathbb{P}-$a.s. We say that π is an optimal *hedging* strategy for the contingent claim G.

The existence of an optimal hedging strategy will be obtained by means of some representation result which is now known as the *optional decomposition theorem* (in the framework of these notes, we can even call it a *predictable decomposition theorem*). As a by-product of this existence result, we will obtain a general dual formulation of the control problem $V(0)$. This will be developed in section 3.
2. In section 4, we seek for more information on the optimal hedging strategy by focusing on the Markov case. The main result is a characterization of $V(0)$ by means of a nonlinear partial differential equation (PDE) with appropriate terminal condition. In some cases, we will be able to solve explicitly the problem. The solution is typically of the *face lifting* type, a desirable property from the viewpoint of the practioners.

The derivation of the above-mentioned PDE is obtained from the dual formulation of $V(0)$ by classical arguments.
3. Section 6 develops the important observation that the same PDE can be obtained by working directly on the original formulation of the problem $V(0)$. Further developments of this idea are reported in [23], [24] and [25]. In particular, such a direct treatment of the problem allows to solve some super-replication problems for which the dual formulation is not available.
4. The final section of this paper is devoted to the problem of super-replication under Gamma constraints, for which no dual formulation is available in the literature. The solution is again of the *face lifting* type.

3 Existence of Optimal Hedging Strategies and Dual Formulation

In this section, we concentrate on the duality approach to the problem of super-replication under portfolio constraints $V(0)$. Our main objective is to convince the reader that the presence of constraints does not affect the general methodology of the proof: the main ingredient is a stochastic representation theorem. We therefore start by recalling the solution in the unconstrained case. This corresponds to the so-called *complete market* framework. In the general constrained case, the proof relies on the same arguments except that: we need to use a more advanced stochastic representation result, namely *the optional decomposition theorem*.

Remark 2. Local Martingale Representation Theorem.
(i) Theorem. Let Y be a local $\mathbb{P}-$local martingale. Then there exits an $I\!R^d-$valued process ϕ such that

$$Y(t) \;=\; Y(0) + \int_0^t \phi(r)'dB(r) \;\; 0 \le t \le T \text{ and } \int_0^T |\phi|^2 \;<\; \infty \;\; \mathbb{P}-\text{a.s.}$$

(see e.g. Dellacherie and Meyer VIII 62).
(ii) We shall frequently need to apply the above theorem to a $\mathbb{Q}-$local martingale Y, for some equivalent probability measure $\mathbb{Q}$ defined by the density $(d\mathbb{Q}/d\mathbb{P}) = Z(T)$ $:= \mathcal{E}\left(-\int_0^T \lambda(r)'dB(r)\right)$, with Brownian motion $B^{\mathbb{Q}} := B + \int_0^\cdot \lambda(r)dr$. To do this, we first apply the local martingale representation theorem to the$\mathbb{P}-$local martingale ZY. The result is $ZY = Y(0) + \int_0^\cdot \phi dB$ for some adapted process ϕ with $\int_0^T |\phi|^2 < \infty$. Applying Itô's lemma, one can easily check that we have:

$$Y(t) \;=\; Y(0) + \int_0^t \psi(r)'dB^{\mathbb{Q}}(r) \;\; 0 \le t \le T \text{ where } \psi \;:=\; Z^{-1}\phi + \lambda Y \;.$$

Since Z and Y are continuous processes on the compact interval $[0,T]$, it is immediately checked that $\int_0^T |\psi|^2 < \infty$ $\mathbb{Q}-$a.s.

3.1 Complete Market: the Unconstrained Black-Scholes World

In this paragraph, we consider the unconstrained case $K = I\!R^d$. The following result shows that $V(0)$ is obtained by the same rule than in the celebrated Black-Scholes model, which was first developed in the case of constant coefficients μ and σ.

Theorem 1. *Assume that $G > 0$ $\mathbb{P}-$a.s. Then:*
(i) $V(0) = \mathbb{E}_0[G]$
(ii) *if* $\mathbb{E}_0[G] < \infty$, *then* $W^\pi_{V(0)}(T) = G$ $\mathbb{P}-$*a.s. for some* $\pi \in \mathcal{A}$.

Proof. 1. Set $F := \{w \in I\!R : W^\pi_w(T) \ge G \text{ for some } \pi \in \mathcal{A}\}$. From the $\mathbb{P}_0-$supermartingale property of the wealth process (3), it follows that $w \ge \mathbb{E}_0[G]$ for all $w \in F$. This proves that $V(0) \ge \mathbb{E}_0[G]$. Observe that this concludes the proof of (i) in the case $\mathbb{E}_0[G] = +\infty$.
2. We then concentrate on the case $\mathbb{E}_0[G] < \infty$. Define

$$Y(t) \;:=\; \mathbb{E}_0[G|\mathcal{F}(t)] \text{ for } 0 \le t \le T \;.$$

Apply the local martingale representation theorem to the $\mathbb{P}_0-$martingale Y, see Remark 2. This provides

$$Y(t) \;=\; Y(0) + \int_0^t \psi(r)'dB_0(r) \text{ for some process } \psi \text{ with } \int_0^T |\psi|^2 < \infty \;.$$

Now set $\pi := (Y\sigma')^{-1}\psi$. Since Y is a positive continuous process, it follows from Standing Assumption that $\pi \in \mathcal{A}$, and $Y = Y(0)\mathcal{E}\left(\int_0^\cdot \pi(r)^*\sigma(r)dB_0(r)\right) = W^\pi_{Y(0)}$. The statement of the theorem follows from the observation that $Y(T) = G$. □

Remark 3. Statement (ii) in the above theorem implies that existence holds for the control problem $V(0)$, i.e. there exists an optimal trading strategy. But it provides a further information, namely that the optimal hedging strategy allows to *attain* the contingent claim G. Hence, in the unconstrained setting, all (positive) contingent claims are attainable. This is the reason for calling this financial market *complete*.

Remark 4. The proof of Theorem 1 suggests that the optimal hedging strategy π is such that the $\mathbb{P}_0-$ martingale Y has the stochastic representation $Y = \mathbb{E}[G] + \int_0^{\cdot} Y\pi'\sigma dB_0$. In the Markov case, we have $Y(t) = v(t, S(t))$. Assuming that v is smooth, it follows from an easy application of Itô's lemma that

$$\Delta^i(t) \ := \ \frac{\pi^i(t) W^{\pi}_{V(0)}(t)}{S^i(t)} \ = \ \frac{\partial v}{\partial s^i}(t, S(t)) \ .$$

We now focus on the positivity condition in the statement of Theorem 1, which rules out the main example of contingent claims, namely European call options $[S^i(T) - K]^+$, and European put options $[K - S^i(T)]^+$. Indeed, since the portfolio process is defined in terms of proportion of wealth, the implied wealth process is strictly positive. Then, it is clear that such contingent claims can not be attained, in the sense of Remark 3, and there is no hope for Claim (ii) of Theorem 1 to hold in this context. However, we have the following easy consequence.

Corollary 1. *Let G be a non-negative contingent claim. Then*
(i) *For all $\varepsilon > 0$, there exists an investment strategy $\pi_\varepsilon \in \mathcal{A}$ such that $W^{\pi_\varepsilon}_{V(0)}(T) = G + \varepsilon$.*
(ii) $V(0) = \mathbb{E}_0[G]$.

Proof. Statement (i) follows from the application of Theorem 1 to the contingent claim $G + \varepsilon$. Now let $V_\varepsilon(0)$ denote the value of the super-replication problem for the contingent claim $G + \varepsilon$. Clearly, $V(0) \le V_\varepsilon(0) = \mathbb{E}_0[G + \varepsilon]$, and therefore $V(0) \le \mathbb{E}_0[G]$ by sending ε to zero. The reverse inequality holds since Part 1 of the proof of Theorem 1 does not require the positivity of G. □

Remark 5. In the Markov setting of Remark 4 above, and assuming that v is smooth, the approximate optimal hedging strategy of Corollary 1 (i) is given by

$$\Delta^i_\varepsilon(t) \ := \ \frac{\pi^i_\varepsilon(t) W^{\pi}_{V_\varepsilon(0)}(t)}{S^i(t)} = \frac{\partial}{\partial s^i}\left\{v(t, S(t)) + \varepsilon\right\} \ = \ \frac{\partial v}{\partial s^i}(t, S(t)) \ ;$$

observe that $\Delta := \Delta_\varepsilon$ is independent of ε.

Example 4. The Black and Scholes formula: consider a financial market with a single risky asset $d = 1$, and let μ and σ be constant coefficients, so that the $\mathbb{P}_0-$distribution of $\ln[S(T)/S(t)]$, conditionally on $\mathcal{F}(t)$, is gaussian with mean $-\sigma^2(T-t)/2$ and variance $\sigma^2(T-t)$. As a contingent claim, we consider the example of a European call option, i.e. $G = [S(T) - K]^+$ for some exercise price $K > 0$. Then, one can compute directly that:

$$V(t) \;=\; v(t,S(t)) \text{ where } v(t,s) \;:=\; sF\left(d(t,s)\right) - KF\left(d(t,s)-\sigma\sqrt{T-t}\right)\;,$$
$$d(t,s) \;:=\; (\sigma\sqrt{T-t})^{-1}\ln(K^{-1}s)+\frac{1}{2}\sigma\sqrt{T-t}\;,$$

and $F(x) = (2\pi)^{-1/2}\int_{-\infty}^{x} e^{-u^2/2}du$ is the cumulative function of the gaussian distribution. According to Remark 4, the optimal hedging strategy in terms of number of shares is given by:

$$\Delta(t) = F\left(d(t,S(t))\right)\;.$$

3.2 Optional Decomposition Theorem

We now turn to the general constrained case. The key-point in the proof of Theorem 1 was the representation of the $\mathbb{P}_0-$martingale Y as a stochastic integral with respect to B_0; the integrand in this representation was then identified to the investment strategy. In the constrained case, the investment strategy needs to be valued in the closed convex set K, which is not guaranteed by the representation theorem. We then need to use a *more advanced* representation theorem. The results of this section were first obtained by ElKaroui and Quenez (1995) for the incomplete market case, and further extended by Cvitanić and Karatzas (1993).

We first need to introduce some notations. Let

$$\delta(y) \;:=\; \sup_{x\in K}\; x'y$$

be the support function of the closed convex set K. Since K contains the origin, δ is non-negative. We shall denote by

$$\tilde{K} \;:=\; \mathrm{dom}(K) = \{y\in I\!R^d \;:\; \delta(y)<\infty\}$$

the effective domain of δ. For later use, observe that $\tilde{K}$ is a closed convex cone of $I\!R^d$. Recall also that, since K is closed and convex, we have the following classical result from convex analysis (see e.g. Rockafellar 1970):

$$x \;\in\; K \text{ if and only if } \delta(y)-x'y \;\geq\; 0 \text{ for all } y\in\tilde{K}\;, \tag{5}$$

$$x \;\in\; \mathrm{ri}(K) \text{ if and only if } x\in K \text{ and } \inf_{y\in\tilde{K}_1}\;(\delta(y)-x'y)\;>\;0\;, \tag{6}$$

where

$$\tilde{K}_1 := \tilde{K}\cap\{y\in I\!R^d \;:\; |y|=1 \text{ and } \delta(y)+\delta(-y)\neq 0\}\;.$$

We next denote by $\mathcal{D}$ the collection of all bounded adapted processes valued in $\tilde{K}$. For each $\nu\in\mathcal{D}$, we set

$$\beta_\nu(t) := \exp\left(-\int_0^t \delta(\nu(r))dr\right)\;,\quad 0\leq t\leq T\;,$$

and we introduce the Doléans-Dade exponential

$$Z_\nu(t) \ := \ \mathcal{E}\left(-\int_0^t \lambda_\nu(r)' dB(r)\right) \text{ where } \lambda_\nu \ := \ \sigma^{-1}(b-\nu) \ = \ \lambda_0 - \sigma^{-1}\nu \ .$$

Since b and ν are bounded, λ_ν inherits the Novikov condition $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T |\lambda_\nu|^2\right)\right] < \infty$ from Standing Assumption. We then introduce the family of probability measures

$$\mathbb{P}_\nu(A) \ := \ \mathbb{E}\left[Z_\nu(t)\mathbf{1}_A\right] \text{ for all } A \in \mathcal{F}(t) \ , \quad 0 \le t \le T \ .$$

Clearly $\mathbb{P}_\nu$ is equivalent to the original probability measure $\mathbb{P}$. By Girsanov Theorem, the process

$$B_\nu(t) := B(t) + \int_0^t \lambda_\nu(r)dr \ = \ B_0(t) - \int_0^t \sigma(r)^{-1}\nu(r)dr \ , \quad 0 \le t \le T \ , \tag{7}$$

is a standard Brownian motion under $\mathbb{P}_\nu$.

Remark 6. The reason for introducing these objects is that the important property (3) extends to the family $\mathcal{D}$:

$$\beta_\nu W_w^\pi \text{ is a } \mathbb{P}_\nu\text{−supermartingale for all } \nu \in \mathcal{D}, \ \pi \in \mathcal{A}_K \ , \tag{8}$$

and $w > 0$. Indeed, by Itô's lemma together with (7),

$$d(W_w^\pi \beta_\nu) = W_w^\pi \beta_\nu \left[-(\delta(\nu) - \pi'\nu)dt + \pi'\sigma dB_\nu\right] \ .$$

In view of (5), this shows that $W_w^\pi \beta_\nu$ is a non-negative local $\mathbb{P}_\nu$−supermartingale, which provides (8).

Theorem 2. *Let Y be an $\mathbb{F}$− adapted positive càdlàg process. Assume that*

the process $\beta_\nu Y$ is a $\mathbb{P}_\nu$−supermartingale for all $\nu \in \mathcal{D}$.

Then, there exists a predictable non-decreasing process C, with $C(0) = 0$, and a constrained portfolio $\pi \in \mathcal{A}_K$ such that $Y = W_{Y(0)}^\pi - C$.

Proof. We start by applying the Doob (unique) decomposition theorem (see e.g. Dellacherie and Meyer VII 12) to the $\mathbb{P}_0$−supermartingale $Y\beta_0 = Y$, together with the local martingle representation theorem, under the probability measure $\mathbb{P}_0$. This implies the existence of an adapted process ψ_0 and a non-decreasing predictable process C_0 satisfying $C_0(0) = 0$, $\int_0^T |\psi_0|^2 < \infty$, and:

$$Y(t) = Y(0) + \int_0^t \psi_0(r)dB_0(r) - C_0(t) \ , \tag{9}$$

see Remark 2. Observe that

$$M_0 \ := \ Y(0) + \int_0^{\cdot} \psi_0 dB_0 = Y + C_0 \ \ge \ Y \ > \ 0 \ . \tag{10}$$

We then define

$$\pi_0 := M_0^{-1}(\sigma')^{-1}\psi_0 \ .$$

From Standing Assumption together with the continuity of M_0 on $[0,T]$ and the fact that $\int_0^T |\psi_0|^2 < \infty$, it follows that $\pi_0 \in \mathcal{A}$. Then $M_0 = W^{\pi}_{Y(0)}$ and by (10),

$$Y = W^{\pi_0}_{Y(0)} - C_0 \ .$$

In order to conclude the proof, it remains to show that the process π is valued in K.
2. By Itô's lemma together with (7), it follows that:

$$d(Y\beta_\nu) = M_0\beta_\nu\pi_0'\sigma dB_\nu - \beta_\nu\left[(Y\delta(\nu) - M_0\pi_0'\nu)dt + dC_0\right] \ .$$

This provides the unique decomposition of the $\mathbb{P}_\nu-$supermartingale $Y\beta_\nu = M_\nu + C_\nu$, with

$$M_\nu \ := \ Y(0) + \int_0^\cdot M_0\beta_\nu\pi_0'\sigma dB_\nu$$

and

$$C_\nu \ := \ \int_0^\cdot \beta_\nu\left[(Y\delta(\nu) - M_0\pi_0'\nu)dt + dC_0\right] \ ,$$

into a $\mathbb{P}_\nu-$local martingale M_ν and a predictable non-decreasing process C_ν starting from the origin. We conclude from this that:

$$\begin{aligned} 0 \ \leq \ \int_0^t \beta_\nu^{-1}dC_\nu &= C_0(t) + \int_0^t \left(Y\delta(\nu) - M_0\pi_0'\nu\right)(r)dr \\ &\leq C_0(t) + \int_0^t M_0\left(\delta(\nu) - \pi_0'\nu\right)(r)dr \ \text{ for all } \nu \in \mathcal{D} \ , \quad (11) \end{aligned}$$

where the last inequality follows from (10) and the non-negativity of the support function δ.
3. Now fix some $\nu \in \mathcal{D}$, and define the set $F_\nu := \{(t,\omega) : [\pi_0'\nu + \delta(\nu)](t,\omega) < 0\}$. Consider the process

$$\nu^{(n)} = \nu 1_{F_\nu^c} + n\nu 1_{F_\nu} \ , \quad n \in I\!N \ .$$

Clearly, since $\tilde{K}$ is a cone, we have $\nu^{(n)} \in \mathcal{D}$ for all $n \in I\!N$. Writing (11) with $\nu^{(n)}$, we see that, whenever $\ell \otimes \mathbb{P}[F_\nu] > 0$, the left hand-side term converges to $-\infty$ as $n \to \infty$, a contradiction. Hence $\ell \otimes \mathbb{P}[F_\nu] = 0$ for all $\nu \in \mathcal{D}$. From (5), this proves that $\pi \in K$ $\ell \otimes \mathbb{P}-$a.s. □

3.3 Dual Formulation

Let $\mathcal{T}$ be the collection of all stopping times valued in $[0,T]$, and define the family of random variables:

$$Y_\tau \;:=\; \operatorname*{ess\,sup}_{\nu\in\mathcal{D}} \; \mathbb{E}_\nu \left[G\gamma_\nu(\tau,T)|\mathcal{F}(\tau)\right] \;;\;\; \tau \;\in\; \mathcal{T} \text{ where } \gamma_\nu(\tau,T) \;:=\; \frac{\beta_\nu(T)}{\beta_\nu(\tau)} \;,$$

and $\mathbb{E}_\nu[\cdot]$ denotes the expectation operator under $\mathbb{P}_\nu$. The purpose of this section is to prove that $V(0) = Y_0$, and that existence holds for the control problem $V(0)$. As a by-product, we will also see that existence for the control problem Y_0 holds only in very specific situations. These results are stated precisely in Theorem 3. As a main ingredient, their proof requires the following (classical) dynamic programming principle.

Lemma 1. (Dynamic Programming). *Let $\tau \le \theta$ be two stopping times in $\mathcal{T}$. Then:*

$$Y_\tau = \operatorname*{ess\,sup}_{\nu\in\mathcal{D}} \; \mathbb{E}_\nu \left[Y_\theta\gamma_\nu(\tau,\theta)|\mathcal{F}(\tau)\right] \;.$$

Proof. 1. Conditioning by $\mathcal{F}(\theta)$, we see that

$$Y_\tau \le \operatorname*{ess\,sup}_{\nu\in\mathcal{D}} \; \mathbb{E}_\nu \left[\gamma_\nu(\tau,\theta)\mathbb{E}_\nu[G\gamma_\nu(\theta,T)|\mathcal{F}(\theta)]\right] \;\le\; \operatorname*{ess\,sup}_{\nu\in\mathcal{D}} \; \mathbb{E}_\nu \left[\gamma_\nu(\tau,\theta)Y_\theta\right] \;.$$

2. To see that the reverse inequality holds, fix any $\mu \in \mathcal{D}$, and let $\mathcal{D}_{\tau,\theta}(\mu)$ be the subset of $\mathcal{D}$ whose elements coincide with μ on the stochastic interval $[\![\tau,\mu]\!]$. Let $(\nu_k)_k$ be a maximizing sequence of Y_θ, i.e.

$$Y_\theta \;=\; \lim_{k\to\infty} \; J_{\nu_k}(\theta) \text{ where } J_\nu(\theta) \;:=\; \mathbb{E}_\nu[G\gamma_\nu(\theta,T)|\mathcal{F}(\theta)] \;;$$

the existence of such a sequence follows from the fact that the family $\{J_\nu(\theta), \nu \in \mathcal{D}\}$ is directed upward. Also, since $J_\nu(\theta)$ depends on ν only through its realization on the stochastic interval $[\theta,T]$, we can assume that $\nu_k \in \mathcal{D}_{\tau,\theta}(\mu)$. We now compute that

$$Y_\tau \ge \mathbb{E}_{\nu_k} \left[G\gamma_{\nu_k}(\tau,T)|\mathcal{F}(\tau)\right] \;=\; \mathbb{E}_\mu \left[\gamma_\mu(\tau,\theta)J_{\nu_k}(\theta)|\mathcal{F}(\tau)\right] \;,$$

which implies that $Y_\tau \ge \mathbb{E}_\mu \left[\gamma_\mu(\tau,\theta)Y_\theta|\mathcal{F}(\tau)\right]$ by Fatou's lemma. □

Now, observe that we may take the stopping times τ in the definition of the family $\{Y_\tau, \tau \in \mathcal{T}\}$ to be deterministic and thereby obtain a non-negative adapted process $\{Y(t), 0 \le t \le T\}$. A natural question is whether this process is consistent with the family $\{Y_\tau, \tau \in \mathcal{T}\}$ in the sense that $Y_\tau(\omega) = Y(\tau(\omega))$ for a.e. $\omega \in \Omega$. For general control problems, this is a delicate issue. However, in our context, it follows from the above dynamic programming principle that the family $\{Y_\tau, \tau \in \mathcal{T}\}$ satisfies a *supermartingale* property:

$$\mathbb{E}\left[\beta_\nu(\theta)Y_\theta|\mathcal{F}(\tau)\right] \;\le\; \beta_\nu(\tau)Y_\tau \text{ for all } \tau,\theta \in \mathcal{T} \text{ with } \tau \;\le\; \theta \;.$$

By a classical argument, this allows to extract a process Y out of this family, which satisfies the supermartingale property in the usual sense. We only state precisely this technical point, and send the interested reader to ElKaroui (1981), Proposition 2.14, or Karatzas and Shreve (1999) Appendix D.

Corollary 2. *There exists a càdlàg process $Y = \{Y(t), 0 \le t \le T\}$, consistent with the family $\{Y_\tau, \tau \in \mathcal{T}\}$, and such that $Y\beta_\nu$ is a $\mathbb{P}_\nu-$supermartingale for all $\nu \in \mathcal{D}$.*

We are now ready for the main result of this section.

Theorem 3. *Assume that $G > 0$ $\mathbb{P}-$a.s. Then:*
(i) *$V(0) = Y(0)$,*
(ii) *if $Y(0) < \infty$, existence holds for the problem $V(0)$, i.e. $W^{\pi}_{V(0)}(T) \ge G$ $\mathbb{P}-$a.s. for some $\pi \in \mathcal{A}_K$,*
(iii) *existence holds for the problem $Y(0)$ if and only if*

$$W^{\hat\pi}_{V(0)}(T) \;=\; G \text{ and } \beta_{\hat\nu} W^{\hat\pi}_{V(0)} \text{ is a } \mathbb{P}_{\hat\nu}-\text{martingale}$$

for some pair $(\hat\pi, \hat\nu) \in \mathcal{A}_K \times \mathcal{D}$.

Proof. 1. We concentrate on the proof of $Y(0) \ge V(0)$ as the reverse inequality is a direct consequence of (8). The process Y, extracted from the family $\{Y_\tau, \tau \in \mathcal{T}\}$ in Corollary 2, satisfies Condition (i) of the optional decomposition theorem 2. Then $Y = W^{\pi}_{Y(0)} - C$ for some constrained portfolio $\pi \in \mathcal{A}_K$, and some predictable non-decreasing process C with $C(0) = 0$. In particular, $W^{\pi}_{Y(0)}(T) \ge Y(T) = G$. This proves that $Y(0) \ge V(0)$, completing the proof of (i) and (ii).
2. It remains to prove (iii). Suppose that $W^{\hat\pi}_{V(0)}(T) = G$ and $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ is a $\mathbb{P}_{\hat\nu}-$martingale for some pair $(\hat\pi, \hat\nu) \in \mathcal{A}_K \times \mathcal{D}$. Then, by the first part of this proof, $Y(0) = V(0) = \mathbb{E}_{\hat\nu}\left[W^{\hat\pi}_{V(0)}(T)\beta_{\hat\nu}(T)\right] = \mathbb{E}_{\hat\nu}\left[G\beta_{\hat\nu}(T)\right]$, i.e. $\hat\nu$ is a solution of $Y(0)$.

Conversely, assume that $Y(0) = \mathbb{E}_{\hat\nu}[G\beta_{\hat\nu}(T)]$ for some $\hat\nu \in \mathcal{D}$. Let $\hat\pi$ be the solution of $V(0)$, whose existence is established in the first part of this proof. By definition $W^{\hat\pi}_{V(0)}(T) - G \ge 0$. Since $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ is a $\mathbb{P}_{\hat\nu}-$super-martingale, it follows that $\mathbb{E}_{\hat\nu}\left[\beta_{\hat\nu}(W^{\hat\pi}_{V(0)}(T) - G)\right] \le 0$. This proves that $W^{\hat\pi}_{V(0)}(T) - G = 0$ $\mathbb{P}-$a.s. We finally see that the $\mathbb{P}_{\hat\nu}-$super-martingale $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ has constant $\mathbb{P}_{\hat\nu}-$expectation:

$$\begin{aligned} Y(0) &\ge \mathbb{E}_{\hat\nu}\left[\beta_{\hat\nu}(t) W^{\hat\pi}_{V(0)}(t)\right] \\ &\ge \mathbb{E}_{\hat\nu}\left[\mathbb{E}_{\hat\nu}\left(\beta_{\hat\nu}(T) W^{\hat\pi}_{V(0)}(T) \middle| \mathcal{F}(t)\right)\right] \;=\; \mathbb{E}_{\hat\nu}\left[\beta_{\hat\nu}(T) G\right] \;=\; Y(0)\,, \end{aligned}$$

and therefore $\beta_{\hat\nu} W^{\hat\pi}_{V(0)}$ is a $\mathbb{P}_{\hat\nu}-$martingale. □

3.4 Extensions

The results of this section can be extended in many directions.

Simple Large Investor Models

Suppose that the drift coefficient of the price process depends on the investor's strategy, so that the price dynamics are influenced by the action of the investor. More precisely, the price dynamics ar defined by:

$$S^i(0) \;=\; s^i \;,\;\; dS^i(t) \;=\; S^i(t)\left[b^i(t,S(t),\pi(t))dt + \sum_{j=1}^d \sigma^{ij}(t)dB^j(t)\right] \;, \quad (12)$$

where $x \longmapsto x'b(t,s,x)$ is a non-negative concave function for all $(t,s) \in [0,T] \times (0,\infty)^d$. Define the Legendre-Fenchel transform:

$$\tilde{b}(t,s,y) \;:=\; \sup_{x\in K} \; x'\left(b(t,s,x)+y\right) \;\; \text{for } y \in I\!R^d \;,$$

together with the associated effective domain:

$$\tilde{K}(t,s) := \left\{ \; y \in I\!R^d \;:\; \tilde{b}(t,s,y) < \infty \; \right\} \;.$$

Next, we introduce the set $\mathcal{D}$ consisting of all bounded adapted processes ν satisfying $\nu(t) \in \tilde{K}(t,S(t))$ $\ell \otimes \mathbb{P}-$a.s.

For any process $\nu \in \mathcal{D}$, define the equivalent probability measure $\mathbb{P}_\nu$ by the density process

$$Z_\nu(t) := \mathcal{E}\left(\int_0^t [\sigma(t)^{-1}\nu(t)]'dW(t)\right) \;,$$

and the discount process:

$$\beta_\nu(t) := \exp\left(-\int_0^t \tilde{b}(r,S(r),\nu(r))dr\right) \;.$$

Then, the results of the previous section can be extended to this context by defining the dynamic version of the dual problem

$$Y_\tau \;:=\; \operatorname*{ess\,sup}_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[G\gamma_\nu(\tau,T)|\,\mathcal{F}(\tau)\right] \;;\;\; \tau \;\in\; \mathcal{T} \text{ where } \gamma_\nu(\tau,T) \;:=\; \frac{\beta_\nu(T)}{\beta_\nu(\tau)} \;.$$

Important observation: in the above *simple* large investor model, only the drift coefficient in the dynamics of S is influenced by the portfolio π. Unfortunately, the analysis developed in the previous paragraphs does not extend to the *general* large investor problem, where the volatility process is influenced by the action of the investor. This is due to the fact that there is no way to get rid of the dependence of σ on π by proceeding to some equivalent change of measure: it is well-known that the measures induced by diffusions with different diffusion coefficients are singular. In section 6, the general large investor problem will be solved by an alternative technique.

Remark 7. From an economic viewpoint, one may argue that the dynamics of the price process should be influenced by the variation of the portfolio holdings of the investor $d[\pi W]$. This is related to the problem of hedging under *gamma* constraints. The last section of these notes presents some preliminary results in this direction.

Semimartingale Price Processes

All the results of the previous sections extend to the case where the price process S is defined as a semimartingale valued in $(0,\infty)^d$. Let us state the generalization of Theorem 3; we refer the interested reader to Föllmer and Kramkov (1997) for a deep analysis.

The wealth process is defined by

$$W_w^\pi(t) := \mathcal{E}\left(\int_0^t \pi(r)'\text{diag}[S(r)]^{-1}dS(r)\right) ,$$

where diag$[s]$ denotes the diagonal matrix with diagonal components s^i, and the set of admissible portfolio $\mathcal{A}_K$ is the collection of all predictable processes π valued in K for which the above stochastic integral is well-defined.

Let $\mathcal{P}$ be the collection of all probability measures $\mathbb{Q} \sim \mathbb{P}$ such that:

$$W_1^\pi e^{-C} \text{ is a } \mathbb{Q}-\text{local supermartingale for all } \pi \in \mathcal{A} , \tag{13}$$

for some increasing predictable process C (depending on $\mathbb{Q}$). An increasing predictable process $C_\mathbb{Q}$ is called an *upper variation process under* $\mathbb{Q}$ if it satisfies (13) and is minimal with respect to this property. Such a process is shown to exist.

Then, the results of the previous sections can be extended to this context by defining the dynamic version of the dual problem

$$Y_\tau \; := \; \operatorname*{ess\,sup}_{\mathbb{Q}\in\mathcal{P}} \mathbb{E}_\mathbb{Q}\left[G\gamma_\mathbb{Q}(\tau,T)\middle|\,\mathcal{F}(\tau)\right], \;\; \tau\in\mathcal{T}, \;\;\; \gamma_\mathbb{Q}(\tau,T) := e^{C_\mathbb{Q}(\tau)-C_\mathbb{Q}(T)},$$

where $\mathbb{E}_\mathbb{Q}$ denotes the expectation operator under $\mathbb{Q}$.

The main ingredient for this result is an extension of the optional decomposition result of Theorem 2. Observe that the non-decreasing process C in this more general framework is *optional*, and not necessarily predictable. We recall that

- the *optional tribe* is generated by $\{\mathcal{F}_t\}-$adapted processes with càd-làg trajectories;

- a process X is *optional* if the map $(t,\omega) \longmapsto X(t,\omega)$ is measurable with respect to the optional tribe;

- the *predictable tribe* is generated by $\{\mathcal{F}_{t-}\}-$adapted processes with left-continuous trajectories;

- a process X is *predictable* if the map $(t,\omega) \longmapsto X(t,\omega)$ is measurable with respect to the predictable tribe.

4 HJB Equation from the Dual Problem

4.1 Dynamic Programming Equation

In order to characterize further the solution of the super-replication problem, we now focus on the Markov case:

$$b(t) = b(t, S(t)) \text{ and } \sigma(t) = \sigma(t, S(t)) ,$$

where b and σ are now vector and matrix valued functions defined on $[0,T] \times I\!R^d$. In order to guarantee existence of a strong solution to the SDE defining S, we assume that b and σ are Lipschitz functions in the s variable, uniformly in t. We also consider the special case of European contingent claim:

$$G = g\left(S(T)\right) ,$$

where g is a map from $[0,\infty)^d$ into $I\!R_+$. We first extend the definition of the dual problem in order to allow for a moving time origin:

$$v(t,s) := \sup_{\nu \in \mathcal{D}} \mathbb{E}\left[g\left(S_{t,s}(T)\right)\gamma_\nu(t,T)\right] = \sup_{\nu \in \mathcal{D}} \mathbb{E}\left[g\left(S_{t,s}(T)\right)e^{-\int_t^T \delta(\nu(r))dr}\right] .$$

Here, $S_{t,s}$ denotes the unique strong solution of (1) with initial data $S_{t,s}(t) = s$. Notice that, although the control process ν is path dependent, the value function of the above control problem $v(t,s)$ is a function of the current values of the state variables. This is a consequence of the important results of Haussman (1985) and ElKaroui, Nguyen and Jeanblanc (1986).

Let v_* and v^* be respectively the lower and upper semi-continuous envelopes of v:

$$v_*(t,s) := \liminf_{(t',s')\to(t,s)} v(t',s') \text{ and } v^*(t,s) := \limsup_{(t',s')\to(t,s)} v(t',s') .$$

We shall frequently appeal to the infinitesimal generator of the process S under $\mathbb{P}_0$:

$$\mathcal{L}\varphi := \varphi_t + \frac{1}{2}\text{Trace}\left[\text{diag}[s]\sigma\sigma'\text{diag}[s]D^2\varphi\right] ,$$

where diag$[s]$ denotes the diagonal matrix with diagonal components s^i, the t subscript denotes the partial derivative with respect to the t variable, $D\varphi$ is the gradient vector with respect to s, and $D^2\varphi$ is the Hessian matrix with respect to s. We will also make use of the first order differential operator:

$$H^y\varphi := \delta(y)\varphi - y'\text{diag}[s]D\varphi \text{ for } y \in \tilde{K} .$$

An important role will be played by the following *face-lifted* payoff function

$$\hat{g}(s) := \sup_{y \in \tilde{K}} g(se^y)e^{-\delta(y)} \text{ for } s \in I\!R^d_+ , \tag{14}$$

where se^y is the $I\!R^d$ vector with components $s^i e^{y^i}$. We finally recall from (6) the notation $\tilde{K}_1 := \tilde{K} \cap \{|y| = 1 \text{ and } \delta(y) + \delta(-y) \neq 0\}$, and we denote by

$$x_K \text{ the orthogonal projection of } x \text{ on } \text{vect}(K) ,$$

where vect(K) is the vector space generated by K. For instance, when K has non-empty interior, we have $x_K = x$. The main result of this section is the following.

Theorem 4. *Let σ be bounded and suppose that v is locally bounded. Assume further that g is lower semi-continuous, $\hat{g}$ is upper semi-continuous with linear growth. Then:*
(i) *For all* $(t,x) \in [0,T) \times I\!R^d$ *and* $y \in \tilde{K}$*, the function* $r \longmapsto \delta(y)r - \ln\left(v(t, e^{x+ry})\right)$ *is non decreasing; in particular,* $v(t, e^x) = v(t, e^{x_K})$ *for all* $(t,x) \in [0,T) \times I\!R^d$*,*
(ii) v *is a (discontinuous) viscosity solution of*

$$\min\left\{-\mathcal{L}v \ , \ \inf_{y \in \tilde{K}_1} \ H^y v\right\} \ = \ 0 \text{ on } [0,T) \times (0,\infty)^d \ , \ \ v(T,\cdot) \ = \ \hat{g} \ . \quad (15)$$

The proof of the above statement is a direct consequence of Propositions 1, 2, 4, 5, and Corollary 3, reported in the following paragraphs.

We conclude this paragraph by recalling the notion of discontinuous viscosity solutions. The interested reader can find an overview of this theory in [7] or [15]. Given a real-valued function u, we shall denote by u_* and u^* its lower and the upper semi-continuous envelopes. We also denote by $\mathcal{S}_n$ the set of all $n \times n$ symmetric matrices with real coefficients.

Definition 2. *Let F be a map from $I\!R^n \times I\!R^n \times \mathcal{S}_n$ into $I\!R$, $\mathcal{O}$ an open domain of $I\!R^n$, and consider the non-linear PDE*

$$F\left(x, u(x), Du(x), D^2u(x)\right) \ = \ 0 \text{ for } x \ \in \ \mathcal{O} \ . \quad (16)$$

Assume further that $F(x,r,p,A)$ is non-increasing in A (in the sense of symmetric matrices). Let $u : \mathcal{O} \longrightarrow I\!R$ be a locally bounded function.
(i) We say that u is a (discontinuous) viscosity supersolution of (16) *if for any $x_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ satisfying*

$$(u_* - \varphi)(x_0) = \min_{x \in \mathcal{O}} (u_* - \varphi) \ ,$$

we have

$$F^*\left(x_0, u_*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \geq 0 \ .$$

(ii) We say that u is a (discontinuous) viscosity subsolution of (16) *if for any $x_0 \in \mathcal{O}$ and $\varphi \in C^2(\mathcal{O})$ satisfying*

$$(u^* - \varphi)(x_0) = \max_{x \in \mathcal{O}} (u^* - \varphi) \ ,$$

we have

$$F_*\left(x_0, u^*(x_0), D\varphi(x_0), D^2\varphi(x_0)\right) \leq 0 \ .$$

(iii) We say that u is a (discontinuous) viscosity solution of (16) *if it satisfies the above requirements* (i) *and* (ii).

In these notes the map F defining the PDE (16) will always be continuous. In [23], the case where F is not continuous is frequently met.

4.2 Supersolution Property

We first start by proving that v_* is a viscosity super-solution of the HJB equation (15).

Proposition 1. *Suppose that the value function v is locally bounded. Then v is a (discontinuous) viscosity super-solution of the PDE*

$$\min\left\{-\mathcal{L}v_* \ , \ \inf_{y\in\tilde{K}} \ H^y v_*\right\}(t,s) \geq 0 \ . \tag{17}$$

Proof. Let (t,s) be fixed in $[0,T) \times I\!\!R_+^d$, and consider a smooth test function φ, mapping $[0,T] \times I\!\!R_+^d$ into $I\!\!R$, and satisfying

$$0 \ = \ (v_* - \varphi)(t,s) = \min_{[0,T]\times I\!\!R_+^d} \ (v_* - \varphi) \ . \tag{18}$$

Let $(t_n, s_n)_{n\geq 0}$ be a sequence of elements of $[0,T) \times I\!\!R_+^d$ satisfying

$$v(t_n,s_n) \ \longrightarrow \ v_*(t,s) \text{ as } n\to\infty \ .$$

In view of (18), this implies that:

$$b_n \ := \ v(t_n,s_n) - \varphi(t_n,s_n) \ \longrightarrow \ 0 \text{ as } n\to\infty \ . \tag{19}$$

The starting point of this proof is the dynamic programming principle of Lemma 1 together with the fact that $v \geq v_*$, which provide:

$$\begin{aligned}\varphi(t_n,s_n) + b_n \ &= \ v(t_n,s_n) \geq \mathbb{E}_\nu\left[v\left(\theta_n, S_{t_n,s_n}(\theta)\right)\gamma_\nu(t_n,\theta_n)\right] \\ &\geq \mathbb{E}_\nu\left[v_*\left(\theta_n, S_{t_n,s_n}(\theta)\right)\gamma_\nu(t_n,\theta_n)\right] \text{ for all } \nu\in\mathcal{D} \ ,\end{aligned} \tag{20}$$

where $\theta_n > t_n$ is an arbitrary stopping time, to be fixed later on. In view of (18) we have $v_* \geq \varphi$, and therefore

$$\begin{aligned}0 &\leq b_n + \mathbb{E}_\nu\left[\varphi(t_n,s_n)\gamma_\nu(t_n,t_n) - \varphi\left(\theta_n, S_{t_n,s_n}(\theta_n)\right)\gamma_\nu(t_n,\theta_n)\right] \\ &= b_n - \mathbb{E}_\nu\left[\int_{t_n}^{\theta_n}\gamma_\nu(t_n,r)\left(\mathcal{L}^{\nu(r)}\varphi - H^{\nu(r)}\varphi\right)(r,S_{t_n,s_n}(r))\,dr\right. \\ &\qquad\left. - \int_{t_n}^{\theta_n}\gamma_\nu(t_n,r)\left(\text{diag}[s]D\varphi\right)(r,S_{t_n,s_n}(r))'\,dB_\nu(r)\right] \ ,\end{aligned} \tag{21}$$

by Itô's lemma. Now, fix some large constant M, and define stopping times

$$\theta_n := h_n \wedge \inf\left\{t\geq t_n \ : \ \sum_{i=1}^d \left|\ln\left(S^i_{t_n,s_n}(t)/s^i_n\right)\right| \geq M\right\} \ ,$$

where

$$h_n := |\beta_n|^{1/2}\mathbf{1}_{\{\beta_n\neq 0\}} + n^{-1}\mathbf{1}_{\{\beta_n=0\}} .$$

We now take the process ν to be constant $\nu(r) = y$ for some $y \in \tilde{K}$, divide (21) by h_n, and send n to infinity using the dominated convergence theorem. The result is

$$-\mathcal{L}\varphi(t,s) + H^y\varphi(t,s) \ \geq \ 0 \text{ for all } y \in \tilde{K} . \tag{22}$$

For $y = 0$, this provides $-\mathcal{L}\varphi(t,s) \geq 0$. It remain to prove that $H^y\varphi(t,s) \geq 0$ for all $y \in \tilde{K}$. This is a direct consequence of (22) together with the fact that $\tilde{K}$ is a cone, and H^y is positively homogeneous in y. □

Remark 8. The above proof can be simplified by observing that the value function v inherits the lower semi-continuity of the payoff function g, assumed in Theorem 4. We did not include this simplification in order to highlight the main point of the proof where the lower semi-continuity of g is needed.

We are now in a position to prove statement (i) of Theorem 4.

Corollary 3. *Suppose that the value function v is locally bounded. For fixed $(t,x) \in [0,T) \times I\!R^d$ and $y \in \tilde{K}$, consider the function*

$$h_y \ : \ r \longmapsto \delta(y)r - \ln\left(v_*(t, e^{x+ry})\right) .$$

Then
(i) h_y *is non-decreasing;*
(ii) $v_*(t,e^x) = v_*(t,e^{x_K})$*, where* $x_K := proj_{\mathrm{vect}(K)}(x)$ *is the orthogonal projection of x on the vector space generated by K;*
(iii) *assuming further that g is lower semi-continuous, we have* $v(t,e^x) = v(t,e^{x_K})$.

Proof. (i) From Proposition 1, function v_* is a viscosity supersolution of

$$H^y v_* \ = \ \delta(y)v - y'\mathrm{diag}[s]Dv_* \ \geq \ 0 \text{ for all } y \in \tilde{K} .$$

Consider the change of variable $w(t,x) := \ln\left[v_*(t,e^{x^1},\ldots,e^{x^d})\right]\exp\left(-\delta(y)t\right)$. Then w is a viscosity supersolution of the equation $-w_x \geq 0$, and therefore w is non-increasing. This completes the proof.
(ii) We first observe that $\mathrm{vect}(K)^\perp \subset \tilde{K}$ and $\delta(y) = 0$ for all $y \in \mathrm{vect}(K)^\perp$. Now, since $\hat{y} := x - x_K \in \mathrm{vect}(K)^\perp$, it follows from (i) that

$$\begin{aligned} -\ln v_*(t,e^{x_K}) \ &= \ h_{\hat{y}}(1) \ \geq \ h_{\hat{y}}(0) = -\ln v_*(t,e^x) \\ &= h_{-\hat{y}}(0) \ \geq \ h_{-\hat{y}}(-1) \ = \ -\ln v_*(t,e^{x_K}) . \end{aligned}$$

Now, assuming further that g is lower semi-continuous, it follows from a simple application of Fatou's lemma that v is lower semicontinuous, and (iii) is just a restatement of (ii). □

4.3 Subsolution Property

The purpose of this paragraph is to prove that the value function v is a (discontinuous) viscosity subsolution of the HJB equation (17). Since it has been shown in Proposition 1 that v is a supersolution of this PDE, this will prove that v is a (discontinuous) viscosity solution of the HJB equation (17).

Proposition 2. *Assume that the payoff function g is lower semi-continuous, and the value function v is locally bounded. Then, v is a (discontinuous) viscosity subsolution of the PDE*

$$\min\left\{-\mathcal{L}v^* \ , \ \inf_{y\in\tilde{K}_1} H^y v^*\right\} \leq 0 \ . \tag{23}$$

Proof. Let $\varphi : [0,T] \times I\!R^d \longrightarrow I\!R$ be a smooth function and $(t_0, s_0) \in [0,T) \times I\!R^d$ be such that:

$$0 \ = \ (v^* - \varphi)(t_0,s_0) \ > \ (v^* - \varphi)(t,s) \text{ for all } (t,s) \neq (t_0,s_0) \ , \tag{24}$$

i.e. (t_0, s_0) is a strict global maximizer of $(v^* - \varphi)$ on $[0,T) \times I\!R^d$. Since $v^* > 0$ and $v^*(t, e^x) = v^*(t, e^{x_K})$ by Corollary 3, we may assume that $\varphi > 0$ and $\varphi(t, e^x)$ $= \varphi(t, e^{x_K})$ as well. Observe that this is equivalent to

$$\operatorname{diag}[s](D\varphi/\varphi)(t,s) \ \in \ \operatorname{vect}(K) \text{ for all } (t,s) \in [0,T) \times (0,\infty)^d \ . \tag{25}$$

In order to prove the required result we shall assume that:

$$-\mathcal{L}\varphi(t_0,s_0) \ > \ 0 \text{ and } \inf_{y\in\tilde{K}_1} \ H^y\varphi(t_0,s_0) \ > \ 0 \ . \tag{26}$$

In view of (25), the second inequality in (26) is equivalent to

$$\operatorname{diag}[s_0]\left(D\varphi/\varphi\right)(t_0,s_0) \in \operatorname{ri}(K) \ . \tag{27}$$

Our final goal will be to end up with a contradiction of the dynamic programming principle of Lemma 1.

1. By smoothness of the test function φ, it follows from (26)-(27) that one can find some parameter $\delta > 0$ such that

$$-\mathcal{L}\varphi(t,s) \ \geq \ 0 \ \text{ and } \ \operatorname{diag}[s]\left(D\varphi/\varphi\right)(t,s) \ \in \ K \tag{28}$$

$$\text{for all } \ (t,s) \ \in \ D \ := \ (t_0 - \delta, t_0 + \delta) \times \left(s_0 e^{\delta B}\right) \ , \tag{29}$$

where B is the unit closed ball of $I\!R^1$ and $s_0 e^{\delta B}$ is the collection of all points ξ in $I\!R^d$ such that $|\ln(\xi^i/s_0^i)| < \delta$ for $i = 1, \ldots, d$. We also set

$$\max_{\partial D}\left(\frac{v^*}{\varphi}\right) \ =: \ e^{-2\beta} < 1 \ , \tag{30}$$

where strict inequality follows from (24).

Next, let $(t_n, s_n)_n$ be a sequence in D such that:

$$(t_n, s_n) \longrightarrow (t, s) \text{ and } v(t_n, s_n) \longrightarrow v^*(t, s) . \tag{31}$$

Using the fact that $v \leq v^*$ together with the smoothness of φ, we may assume that the sequence $(t_n, s_n)_n$ satisfies the additional requirement

$$v(t_n, s_n) \geq e^{-\beta}\varphi(t_n, s_n) . \tag{32}$$

For ease of notation, we denote $S_n(\cdot) := S_{t_n,s_n}(\cdot)$, and we introduce the stopping times

$$\theta_n := \inf\{r \geq t_n \ : \ (r, S_n(r)) \notin D\} .$$

Observe that, by continuity of the process S_n,

$$(\theta_n, S_n(\theta_n)) \in \partial D \text{ so that } v^*\left(\theta_n, S_n(\theta_n)\right) \leq e^{-2\beta}\varphi\left(\theta_n, S_n(\theta_n)\right) , \tag{33}$$

as a consequence of (30).
2. Let ν be any control process in $\mathcal{D}$. Since $v \leq v^*$, it follows from (32) and (33) that:

$$\begin{aligned} v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n) - v(t_n, s_n) &\leq e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) \\ &+ e^{-2\beta}\left(\varphi\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n) - \varphi(t_n, s_n)\right) . \end{aligned}$$

We now apply Itô's lemma, and use the definition of θ_n together with (28). The result is:

$$\begin{aligned} & v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n) - v(t_n, s_n) \\ & \quad e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) + e^{-2\beta}\int_{t_n}^{\theta_n} \gamma_\nu(t_n, r)(-H^{\nu(r)} + \mathcal{L})\varphi(r, S_n(r))dr \\ & \quad\quad + e^{-2\beta}\int_{t_n}^{\theta_n} \gamma_\nu(t_n, r)(D\varphi' \text{diag}[s]\sigma)(r, S_n(r))dB_\nu(r) \\ & \quad \leq e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) + e^{-2\beta}\int_{t_n}^{\theta_n} (D\varphi' \text{diag}[s]\sigma)(r, S_n(r))dB_\nu(r) . \end{aligned}$$

We finally take expected values under $\mathbb{P}_\nu$, and use the arbitrariness of $\nu \in \mathcal{D}$ to see that:

$$\begin{aligned} v(t_n, s_n) &\leq \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n)|\mathcal{F}(t_n)\right] + e^{-2\beta}\left(1 - e^{\beta}\right)\varphi(t_n, s_n) \\ &< \sup_{\nu\in\mathcal{D}} \mathbb{E}_\nu\left[v\left(\theta_n, S_n(\theta_n)\right)\gamma_\nu(t_n, \theta_n)|\mathcal{F}(t_n)\right] , \end{aligned}$$

which is in contradiction with the dynamic programming principle of Lemma 1. □

4.4 Terminal Condition

From the definition of the value function v, we have:

$$v(T,s) \;=\; g(s) \text{ for all } s \in I\!R^d_+ \;.$$

However, we are facing a singular stochastic control problem, as the controls ν are valued in an unbounded set. Typically this situation induces a *jump* in the terminal condition so that we only have:

$$v_*(T,s) \geq v(T,s) \;=\; g(s) \;.$$

The main difficulty is then to characterize the terminal condition of interest $v_*(T,\cdot)$. The purpose of this section is to prove that $v_*(T,\cdot)$ is related to the function $\hat{g}$ defined in (14).

We first start by deriving the PDE satisfied by $v_*(T,\cdot)$, as inherited from Proposition 1.

Proposition 3. *Suppose that g is lower semi-continuous and v is locally bounded. Then $v_*(T,\cdot)$ is a viscosity super-solution of*

$$\min\left\{v_* - g \;,\; \inf_{y\in\tilde{K}} \; H^y v_*\right\}(T \geq 0) \;.$$

Proof. 1. We first check that $v_*(T,\cdot) \geq g$. Let $(t_n,s_n)_n$ be a sequence of $[0,T) \times (0,\infty)^d$ converging to (T,s), and satisfying $v(t_n,s_n) \longrightarrow v_*(T,s)$. Since $\delta(0) = 0$, it follows from the definition of v that

$$v(t_n,s_n) \geq \mathbb{E}_0\left[g\left(S_{t_n,s_n}(T)\right)\right] \;.$$

Since $g \geq 0$, we may apply Fatou's lemma, and derive the required inequality using the lower semi-continuity condition on g, together with the continuity of $S_{t,s}(T)$ in (t,s).

2. It remains to prove that $v_*(T,\cdot)$ is a viscosity super-solution of

$$H^y v_*(T,\cdot) \;\geq\; 0 \text{ for all } y \in \tilde{K} \;. \tag{34}$$

Let $f \leq v_*(T,\cdot)$ be a C^2 function satisfying, for some $s_0 \in (0,\infty)^d$,

$$0 \;=\; (v_*(T,\cdot) - f)(s_0) = \min_{I\!R^d_+}\,(v_*(T,\cdot) - f) \;.$$

Since v_* is lower semi-continuous, we have $v_*(T,s_0) = \liminf_{(t,s)\to(T,s_0)}\; v_*(t,s)$, and

$$v_*(T_n,s_n) \;\longrightarrow\; v_*(T,s_0) \text{ for some sequence } \;(T_n,s_n) \;\longrightarrow\; (T,s_0) \;.$$

Define

$$\varphi_n(t,s) := f(s) - \frac{1}{2}|s-s_0|^2 + \frac{T-t}{T-T_n}\ ,$$

let $\bar{B} = \{s \in I\!R^d_+ \ : \ \sum_i |\ln(s^i/s^i_0)| \le 1\}$, and choose $(\bar{t}_n, \bar{s}_n)$ such that:

$$(v_* - \varphi_n)(\bar{t}_n, \bar{s}_n) = \min_{[T_n,T]\times\bar{B}} (v_* - \varphi_n)\ .$$

We shall prove the following claims:

$$\bar{t}_n < T \text{ for large n}\ , \tag{35}$$

$$\bar{s}_n \longrightarrow s_0 \text{ along some subsequence, and } v_*(\bar{t}_n, \bar{s}_n) \longrightarrow v_*(T, s_0)\ . \tag{36}$$

Admitting this, wee see that, for sufficiently large n, $(\bar{t}_n, \bar{s}_n)$ is a local minimizer of the difference $(v_* - \varphi_n)$. Then, the viscosity super-solution property, established in Proposition 1, holds at $(\bar{t}_n, \bar{s}_n)$, implying that $H^y v_*(\bar{t}_n, \bar{s}_n) \ge 0$, i.e.

$$\delta(y)v_*(\bar{t}_n, \bar{s}_n) - y'\mathrm{diag}[s]\left(Df(\bar{s}_n) - (\bar{s}_n - s_0)\right) \ \ge\ 0 \text{ for all } y \in \tilde{K}_1\ ,$$

by definition of φ_n in terms of f. In view of (36), this provides the required inequality (34).

Proof of (35) : Observe that for all $s \in \bar{B}$,

$$(v_* - \varphi_n)(T,s) \ = \ v_*(T,s) - f(s) + \frac{1}{2}|s-s_0|^2 \ge v_*(T,s) - f(s) \ \ge\ 0\ .$$

Then, the required result follows from the fact that:

$$\begin{aligned}\lim_{n\to\infty}(v_* - \varphi_n)(T_n, s_n) &= \lim_{n\to\infty}\left\{v_*(T_n,s_n) - f(s_n) + \frac{1}{2}|s_n - s_0|^2 - \frac{1}{T-T_n}\right\}\\ &= -\infty\ .\end{aligned}$$

Proof of (36): Since $(\bar{s}_n)_n$ is valued in the compact subset $\bar{B}$, we have $\bar{s}_n \longrightarrow \bar{s}$ along some subsequence, for some $\bar{s} \in \bar{B}$. We now use respectively the following facts: s_0 minimizes the difference $v_*(T,\cdot) - f$, v_* is lower semi-continuous, $s_n \longrightarrow s_0$, $\bar{t}_n \ge T_n$, and $(\bar{t}_n, \bar{s}_n)$ minimizes the difference $v_* - \varphi_n$ on $[T_n, T] \times \bar{B}$. The result is:

$$\begin{aligned}0 &\le (v_*(T,\cdot) - f)(\bar{s}) - (v_*(T,\cdot) - f)(s_0)\\ &\le \liminf_{n\to\infty}\ \Big\{(v_* - \varphi_n)(\bar{t}_n, \bar{s}_n) - \frac{1}{2}|\bar{s}_n - s_0|^2\\ &\qquad\qquad -(v_* - \varphi_n)(T_n, s_n) + \frac{1}{2}|s_n - s_0|^2 - \frac{\bar{t}_n - T_n}{T - T_n}\Big\}\\ &\le -\frac{1}{2}|\bar{s} - s_0|^2 + \liminf_{n\to\infty}\ \{(v_* - \varphi_n)(\bar{t}_n, \bar{s}_n) - (v_* - \varphi_n)(T_n, s_n)\}\\ &\le -\frac{1}{2}|\bar{s} - s_0|^2 + \limsup_{n\to\infty}\ \{(v_* - \varphi_n)(\bar{t}_n, \bar{s}_n) - (v_* - \varphi_n)(T_n, s_n)\}\\ &\le -\frac{1}{2}|\bar{s} - s_0|^2 \ \le\ 0\ ,\end{aligned}$$

so that all above inequalities hold with equality, and (36) follows. □

We are now able to characterize a precise bound on the terminal condition of the singular stochastic control problem $v(t,s)$.

Proposition 4. *Suppose that g is lower semi-continuous and v is locally bounded. Then $v_*(T,\cdot) \geq \hat{g}$.*

Proof. We first change variables by setting $F(x) := \ln\left[v_*(T,e^x)\right]$, with $e^x := (e^{x^1},\ldots,e^{x^d})'$. From Proposition 3, it follows that F satisfies:

$$\delta(y) - y'DF \;\geq\; 0 \text{ for all } y \in \tilde{K} \;.$$

Introducing the lower semi-continuous function $h(t) := F(x+ty) - \delta(y)t$, for fixed $x \in \mathbb{R}^d$ and $y \in \tilde{K}$, we see that h is a viscosity super-solution of the equation $-h' \geq 0$. By classical result in the theory of viscosity solutions, we conclude that h is non-increasing. In particular $h(0) \geq h(1)$, i.e. $F(x) \geq F(x+y) - \delta(y)$ for all $x \in \mathbb{R}^d$ and $y \in \tilde{K}$, and

$$\begin{aligned} F(x) &\geq \sup_{y\in\tilde{K}} \;\{F(x+y) - \delta(y)\} \\ &\geq \sup_{y\in\tilde{K}} \;\ln\left\{g\left(e^{x+y}\right)e^{-\delta(y)}\right\} \;\text{ for all }\; x \in \mathbb{R}^d \;. \end{aligned}$$

This provides the required result by simply turning back to the initial variables. □

We now intend to show that the reverse inequality of Proposition 4 holds. In order to simplify the presentation, we shall provide an easy proof which requires strong assumptions.

Proposition 5. *Let σ be a bounded function, and $\hat{g}$ be an upper semi-continuous function with linear growth. Suppose that v is locally bounded. Then $v^*(T,\cdot) \leq \hat{g}$.*

Proof. Suppose to the contrary that $V^*(T,s) - \hat{g}(s) =: 2\eta > 0$ for some $s \in (0,\infty)^d$. Let (T_n, s_n) be a sequence in $[0,T] \times (0,\infty)^d$ satisfying:

$$(T_n, s_n) \;\longrightarrow\; (T,s)\;,\;\; V(T_n,s_n) \longrightarrow\; V^*(T,s)$$

and

$$V(T_n,s_n) \;>\; \hat{g}(s) + \eta \text{ for all } n \geq 1 \;.$$

From the (dual) definition of V, this shows the existence of a sequence $(\nu^n)_n$ in $\mathcal{D}$ such that:

$$\mathbb{E}^0_{T_n,s_n}\left[g\left(S_T^{(n)} e^{\int_{T_n}^T \nu_r^n dr}\right) e^{-\int_{T_n}^T \delta(\nu_r^n)dr}\right] \;>\; \hat{g}(s) + \eta \text{ for all } n \geq 1 \;, \quad (37)$$

where

$$S_T^{(n)} := s_n \mathcal{E}\left(\int_{T_n}^T \sigma(t, S_t^{\nu^n}) dW_t\right) .$$

We now use the sublinearity of δ to see that:

$$\begin{aligned} &\mathbb{E}^0_{T_n,s_n}\left[g\left(S_T^{(n)} e^{\int_{T_n}^T \nu_r^n dr}\right) e^{-\int_{T_n}^T \delta(\nu_r^n) dr}\right] \\ &\quad \le \mathbb{E}^0_{T_n,s_n}\left[g\left(S_T^{(n)} e^{\int_{T_n}^T \nu_r^n dr}\right) e^{-\delta(\int_{T_n}^T \nu_r^n dr)}\right] \\ &\quad \le \mathbb{E}^0_{T_n,s_n}\left[\hat{g}\left(S_T^{(n)}\right)\right] , \end{aligned}$$

where we also used the definition of $\hat{g}$ together with the fact that $\tilde{K}$ is a closed convex cone of $I\!R^d$. Plugging this inequality in (37), we see that

$$\hat{g}(s) + \eta \le \mathbb{E}^0_{t_n,s_n}\left[\hat{g}\left(S_T^{(n)}\right)\right] . \tag{38}$$

By easy computation, it follows from the linear growth condition on $\hat{g}$ that

$$\mathbb{E}^0 \left|\hat{g}(S_T^{(n)})\right|^2 \le Const \left(1 + e^{(T-t)\|\sigma\|_\infty^2}\right) .$$

This shows that the sequence $\left\{\hat{g}(S_T^{(n)}),\ n \ge 1\right\}$ is bounded in $L^2(\mathbb{P}^0)$, and is therefore uniformly integrable. We can therefore pass to the limit in (38) by means of the dominated convergence theorem. The required contradiction follows from the upper semi-continuity of $\hat{g}$ together with the a.s. continuity of S_T in the initial data (t, s). □

5 Applications

5.1 The Black-Scholes Model with Portfolio Constraints

In this paragraph we report an explicit solution of the super-replication problem under portfolio constraints in the context of the Black-Scholes model. This result was obtained by Broadie, Cvitanić and Soner (1997).

Proposition 6. *Let $d = 1$, $\sigma(t, s) = \sigma > 0$, and consider a lower semi-continuous payoff function $g : I\!R_+ \longrightarrow I\!R$. Assume that the face-lifted payoff function $\hat{g}$ is upper semi-continuous and has linear growth. Then:*

$$v(t, s) = \mathbb{E}_0\left[\hat{g}\left(S_{t,s}(T)\right)\right] ,$$

i.e. $v(t, s)$ is the unconstrained Black-and Scholes price of the face-lifted contingent claim $\hat{g}\left(S_{t,s}(T)\right)$.

Sketch of the Proof By a direct application of Theorem 4, the value function $v(t, s)$ solves the PDE (15). When σ is constant, it follows from the maximum principle that the PDE (15) reduces to:

$$-\mathcal{L}v = 0 \text{ on } [0, T) \times (0, \infty)^d , \ v(T, \cdot) = \hat{g} ,$$

and the required result follows from the Feynman-Kac representation formula. □

5.2 The Uncertain Volatility Model

In this paragraph, we study the simplest incomplete market model. The number of risky assets is now $d = 2$. We consider the case $K = I\!R \times \{0\}$ so that the second risky asset is not tradable. In order to satisfy the conditions of Theorem 4, we assume that the contingent claim is defined by $G = g(S^1(T))$, where the payoff function $g : I\!R_+ \longrightarrow I\!R_+$ is continuous and has polynomial growth. We finally introduce the notations:

$$\overline{\sigma}(t, s_1) \; := \; \sup_{s_2>0} \, [\sigma_{11}^2 + \sigma_{12}^2](t, s_1, s_2) \text{ and } \underline{\sigma}(t, s_1) \; := \; \inf_{s_2>0} \, [\sigma_{11}^2 + \sigma_{12}^2](t, s_1, s_2)$$

We report the following result from Cvitanić, Pham and Touzi (1999) which follows from Theorem 4. We leave its verification for the reader.

Proposition 7. (i) *Assume that* $\overline{\sigma} < \infty$ *on* $[0, T] \times I\!R_+$. *Then* $v(t, s) = v(t, s_1)$ *solves the Black-Scholes-Barrenblatt equation*

$$-v_t - \frac{1}{2}\left[\overline{\sigma}^2 v_{s_1 s_1}^+ - \underline{\sigma}^2 v_{s_1 s_1}^-\right] = 0, \text{ on } [0, T) \times (0, \infty), \;\; v(T, s_1) = g(s_1) \text{ for } s_1 > 0.$$

(ii) *Assume that* $\overline{\sigma} = \infty$ *and*

$$\text{either } g \text{ is convex or } \underline{\sigma} \; = \; 0 \; .$$

Then $v(t, s) = g^{conc}(s_1)$, *where* g^{conc} *is the concave envelope of* g.

6 HJB Equation from the Primal Problem for the General Large Investor Problem

In this section, we present an original dynamic programming principle stated directly on the initial formulation of the super-replication problem. We then prove that the HJB equation of Theorem 4 can be obtained by means of this dynamic programming principle. Hence, if one is only interested in the derivation of the HJB equation, then the dual formulation is not needed any more.

An interesting feature of the analysis of this section is that it can be extended to problems where the dual formulation is not available. A first example is the large investor problem with feedback of the investor's portfolio on the price process through the drift and the volatility coefficients; see section 3.4. This problem is treated in the present section and does not present any particular difficulty. Another example is the problem of super-replication under gamma constraints which will be discussed in the last section of these notes. The analysis of this section can be further extended to cover front propagation problems related to the theory of differential geometry, see [23].

6.1 Dynamic Programming Principle

In this section, the price process S is defined by the controlled stochastic differential equation:

$$\begin{aligned} S^{\pi}_{t,s}(t) &= s \\ dS^{\pi}_{t,s}(r) &= \mathrm{diag}[S^{\pi}_{t,s}](r)\left[b(r,S^{\pi}_{t,s}(r),\pi(r))dr + \sigma(r,S^{\pi}_{t,s}(r),\pi(r))dB(r)\right] \ , \end{aligned}$$

where $b:[0,T]\times I\!R^d\times K\longrightarrow I\!R^d$ and $\sigma:[0,T]\times I\!R^d\times K\longrightarrow I\!R^{d\times d}$ are bounded functions, Lipschitz in the s variable uniformly in (t,π).

The wealth process is defined by

$$\begin{aligned} W^{\pi}_{t,s,w}(t) &= w \\ dW^{\pi}_{t,s,w}(r) &= W^{\pi}_{t,s,w}(r)\pi(r)'\left[b(r,S^{\pi}_{t,s}(r),\pi(r))dr + \sigma(r,S^{\pi}_{t,s}(r),\pi(r))dB(r)\right] \ . \end{aligned}$$

Given a European contingent claim $G = g(S_{t,s}(T)) \geq 0$, the super-replication problem is defined by

$$v(t,s) := \inf\left\{w>0 \ : \ W^{\pi}_{t,s,w}(T) \ \geq \ g\left(S^{\pi}_{t,s}(T)\right) \ \mathbb{P}-\text{a.s. for some } \pi\in\mathcal{A}_K\right\} \ .$$

The main result of this section is the following.

Lemma 2. *Let* $(t,s)\in[0,T)\times(0,\infty)^d$ *be fixed, and consider an arbitrary stopping time* θ *valued in* $[t,T]$. *Then,*

$$v(t,s) = \inf\left\{w>0 \ : \ W^{\pi}_{t,s,w}(\theta) \ \geq \ v\left(\theta,S^{\pi}_{t,s}(\theta)\right) \ \mathbb{P}-\textit{a.s. for some } \pi\in\mathcal{A}_K\right\} \ .$$

In order to derive the HJB equation by means of the above dynamic programming principle, we shall write it in the following equivalent form:

DP1 Let $(t,s)\in[0,T)\times(0,\infty)^d$, $(w,\pi)\in(0,\infty)\times\mathcal{A}_K$ be such that $W^{\pi}_{t,s,w}(T) \geq g\left(S^{\pi}_{t,s}(T)\right)$ $\mathbb{P}-$a.s. Then

$$W^{\pi}_{t,s,w}(\theta) \ \geq \ v\left(\theta,S^{\pi}_{t,s}(\theta)\right) \ \ \mathbb{P}-\text{a.s.}$$

DP2 Let $(t,s)\in[0,T)\times(0,\infty)^d$, and set $\hat{w}:=v(t,s)$. then for all stopping time θ valued in $[t,T]$,

$$\mathbb{P}\left[W^{\pi}_{t,s,\hat{w}-\eta}(\theta) \ > \ v\left(\theta,S^{\pi}_{t,s}(\theta)\right)\right] \ < \ 1 \text{ for all } \eta>0 \text{ and } \pi\in\mathcal{A}_K \ .$$

Idea of the proof. The proof of DP1 follows from the trivial observation that

$$S^{\pi}_{t,s}(T) \ = \ S^{\pi}_{\theta,S^{\pi}_{t,s}(\theta)}(T) \text{ and } W^{\pi}_{t,s,w}(T) \ = \ W^{\pi}_{\theta,S_{t,s}(\theta),W_{t,s,w}(\theta)}(T) \ .$$

The proof of DP2 is more technical, we only outline the main idea and send the interested reader to [23]. Let θ be some stopping time valued in $[t,T]$, and suppose that

$$\hat{W} \ := \ W^{\pi}_{t,s,\hat{w}-\eta}(\theta) \ > \ v\left(\theta,S^{\pi}_{t,s}(\theta)\right) \ \mathbb{P}-\text{a.s. for some } \eta>0 \text{ and } \pi\in\mathcal{A}_K \ .$$

Set $\hat{S} := S^{\pi}_{t,s}(\theta)$. By definition of the super-replication problem $v\left(\theta, \hat{S}\right)$ starting at time θ, there exists an admissible portfolio $\hat{\pi} \in \mathcal{A}_K$ such that

$$W^{\hat{\pi}}_{\theta,\hat{S},\hat{W}}(T) \geq g\left(S^{\hat{\pi}}_{\theta,\hat{S}}(T)\right) \;\; \mathbb{P}-\text{a.s.}$$

This is the delicate place of this proof, as one has to appeal to a measurable selection argument in order to define the portfolio $\hat{\pi}$. In order to complete the proof, it suffices to define the admissible portfolio $\tilde{\pi} := \pi \mathbf{1}_{[t,\theta)} + \hat{\pi}\mathbf{1}_{[\![\theta,T]\!]}$, and to observe that:

$$W^{\tilde{\pi}}_{t,s,\hat{w}-\eta}(T) \;=\; W^{\hat{\pi}}_{\theta,\hat{S},\hat{W}}(T) \geq g\left(S^{\hat{\pi}}_{\theta,\hat{S}}(T)\right) \;=\; g\left(S^{\tilde{\pi}}_{t,s}(T)\right) \;\; \mathbb{P}-\text{a.s.}$$

which is in contradiction with the definition of $\hat{w}$. □

6.2 Supersolution Property from DP1

In this paragraph, we prove the following result.

Proposition 8. *Suppose that the value function v is locally bounded, and the constraint set K is compact. Then v is a discontinuous viscosity supersolution of the PDE*

$$\min\left\{ -\hat{\mathcal{L}}v_* \;,\; \inf_{y\in\tilde{K}} \; H^y v_* \right\}(t,s) \geq 0$$

where

$$\hat{\mathcal{L}}u := u_t(t,s) + \frac{1}{2}\text{Tr}\left[\text{diag}[s]\sigma\sigma'(t,s,\hat{\alpha})\text{diag}[s]D^2u(t,s)\right]$$
$$\hat{\alpha} := \text{diag}[s]\frac{Du}{u}(t,s) \;.$$

The compactness condition on the constraints set K is assumed in order to simplify the proof, see [26]. In the case of a *small* investor model (b and σ do not depend on π), the proof of Proposition 8 is an alternative proof of Proposition 1 which uses part DP1 of the above direct dynamic programming principle of Lemma 2, instead of the classical dynamic programming principle of Lemma 1 stated on the dual formulation of the problem. Recall that such a dual formulation is not available in the context of a general large investor model with σ depending on π.

Proof. Let (t,s) be fixed in $[0,T) \times I\!R^d_+$, and consider a smooth test function φ, mapping $[0,T] \times I\!R^d_+$ into $(0,\infty)$, and satisfying

$$0 \;=\; (v_* - \varphi)(t,s) = \min_{[0,T]\times I\!R^d_+} \; (v_* - \varphi) \;. \tag{39}$$

Let $(t_n, s_n)_{n\geq 0}$ be a sequence of elements of $[0,T) \times I\!R^d_+$ satisfying

$$v(t_n, s_n) \;\longrightarrow\; v_*(t,s) \text{ as } n \to \infty \;.$$

We now set

$$w_n := v(t_n, s_n) + \max\{n^{-1}\ ,\ 2|v(t_n, s_n) - \varphi(t_n, s_n)|\}\ \text{ for all } n \geq 0\ ,$$

and observe that

$$c_n\ :=\ \ln w_n - \ln \varphi(t_n, s_n)\ >\ 0 \text{ and } c_n\ \longrightarrow\ 0 \text{ as } n \to \infty\ , \tag{40}$$

in view of (39). Since $w_n > v(t_n, s_n)$, it follows from DP1 that:

$$\ln\left\{v\left(\theta_n, S^{\pi_n}_{t_n,s_n}(\theta_n)\right)\right\}\ \leq\ \ln\left\{W^{\pi_n}_{t_n,s_n,w_n}(\theta_n)\right\} \text{ for some } \pi_n \in \mathcal{A}_K\ ,$$

where, for each $n \geq 0$, θ_n is an arbitrary stopping time valued in $[t_n, T]$, to be chosen later on. In the rest of this proof, we simply denote $(S_n, W_n) := \left(S^{\pi_n}_{t_n,s_n}, W^{\pi_n}_{t_n,s_n,w_n}\right)$.

By definition of the test function φ, we have $v \geq v_* \geq \varphi$, and therefore:

$$\begin{aligned}0 \leq\ & c_n + \ln \varphi(t_n, s_n) - \ln \varphi\left(\theta_n, S_n(\theta_n)\right)\\ &+ \int_{t_n}^{\theta_n} \left[\pi_n' b - \frac{1}{2}\left|\sigma'\pi_n\right|^2\right](r, S_n(r), \pi_n(r))\, dr\\ &+ \int_{t_n}^{\theta_n} \left(\pi_n'\sigma\right)(r, S_n(r), \pi_n(r))\, dB(r)\ .\end{aligned}$$

Applying Itô's lemma to the smooth test function $\ln\varphi$, we see that:

$$\begin{aligned}0 \leq c_n &+ \int_{t_n}^{\theta_n} (\pi_n - h_n)'\sigma\left(r, S_n(r), \pi_n(r)\right) dB(r)\\ &- \int_{t_n}^{\theta_n} \left[\frac{\mathcal{L}^{\pi_n(r)}\varphi}{\varphi} - (\pi_n - h_n)'b + \frac{|\sigma' h_n|^2 - |\sigma'\pi_n|^2}{2}\right](r, S_n(r), \pi_n(r))\, dr\ ,\end{aligned} \tag{41}$$

where we introduced the additional notations

$$h_n(r) := \text{diag}[S_n(r)]\frac{D\varphi}{\varphi}\ (r, S_n(r))\ ,$$

and

$$\mathcal{L}^\alpha\varphi(t, s) := \varphi_t(t, s) + \frac{1}{2}\text{Tr}\left\{\text{diag}[s]\sigma\sigma'(t, s, \alpha)\text{diag}[s]D^2\varphi(t, s)\right\}\ .$$

2. Given a positive integer k, we define the equivalent probability measure $\mathbb{P}^k_n$ by:

$$\left.\frac{d\mathbb{P}^k_n}{d\mathbb{P}}\right|_{\mathcal{F}(t_n)} := \mathcal{E}\left(\int_{t_n}^T \sigma^{-1}\left[-b + k\frac{\pi_n - h_n}{|\pi_n - h_n|}1_{\{\pi_n - h_n \neq 0\}}\right](r, S_n(r))\, dB(r)\right)\ ;$$

recall that σ^{-1} and b are bounded functions. Taking expected values in (41) under $\mathbb{P}^k_n$, conditionally on $\mathcal{F}(t_n)$, we see that:

$$0 \le c_n - \mathbb{E}_k \int_{t_n}^{\theta_n} \left[\frac{\mathcal{L}^{\pi_n(r)}\varphi}{\varphi} + k|\pi_n - h_n| + \frac{|\sigma' h_n|^2 - |\sigma'\pi_n|^2}{2} \right] (r, S_n(r), \pi_n(r))\, dr \ ,$$

where $\mathbb{E}_k$ denote the conditional expectation operator under $\mathbb{P}_n^k$. We now divide by $\sqrt{c_n}$ and send n to infinity to get:

$$0 \ \le \ \liminf_n \mathbb{E}_k \int_{t_n}^{\theta_n} \left[\frac{-\mathcal{L}^{\pi_n(r)}\varphi}{\varphi} - k|\pi_n - h_n| - \frac{|\sigma' h_n|^2 - |\sigma'\pi_n|^2}{2} \right] (r, S_n(r), \pi_n(r))\, dr \ , \tag{42}$$

3. We now fix the stopping times:

$$\theta_n := \sqrt{c_n} \wedge \inf\left\{ r \ge t_n \ : \ S_n(r) \notin s_n e^{\gamma B} \right\} \ ,$$

for some constant γ, where B is the unit closed ball of $I\!R^1$ and $s_n e^{\delta B}$ is the collection of all points ξ in $I\!R^d$ such that $|\ln(\xi^i/s_n^i)| \le \gamma$. Then, by dominated convergence, it follows from (42) that:

$$0 \le \inf_{\alpha \in K} \ -\frac{\mathcal{L}^\alpha \varphi}{\varphi} - k\left|\alpha - \text{diag}[s]\frac{D\varphi}{\varphi}(t,s)\right| + \frac{1}{2}\left[\left|\sigma' \text{diag}[s]\frac{D\varphi}{\varphi}(t,s)\right|^2 - |\sigma'\alpha|^2 \right] \ .$$

Since K is compact, the above minimum is attained at some $\alpha_k \in K$, and $\alpha_k \longrightarrow \hat{\alpha} \in K$ along some subsequence. Sending k to infinity, we see that one must have

$$\hat{\alpha} \ = \ \text{diag}[s]\frac{D\varphi}{\varphi}(t,s) \ \in \ K \text{ and } -\mathcal{L}^{\hat{\alpha}}\varphi \ \ge \ 0\ .\square$$

6.3 Subsolution Property from DP2

We now show that part DP2 of the dynamic programming principle stated in Lemma 2 allows to prove an extension of the subsolution property of Proposition 2 in the general large investor model. To simplify the presentation, we shall only work out the proof in the case where K has non-empty interior; the general case can be addressed by the same argument as in section 4.3, i.e. by first deducing, from the supersolution property, that $v(t, e^x)$ depends only on the projection x_K of x on $\text{aff}(K) = \text{vect}(K)$.

Proposition 9. *Suppose that the value function v is locally bounded, and the constraints set K has non-empty interior. Then, v is a discontinuous viscosity subsolution of the PDE*

$$\min\left\{ -\hat{\mathcal{L}}v^* \ , \ \inf_{y \in \tilde{K}_1} \ H^y v^* \right\}(t,s) \le 0$$

Proof. 1. In order to simplify the presentation, we shall pass to the log-variables. Set $x := \ln w$, $X^\pi_{t,s,x} := \ln W^\pi_{t,s,w}$, and $u := \ln v$. By Itô's lemma, the controlled process X^π is given by:

$$X^{\pi}_{t,s,x}(\tau) = x + \int_t^{\tau} \left(\pi' b - \frac{1}{2}|\sigma'\pi|^2 \right) \left(r, S^{\pi}_{t,s}(r), \pi(r) \right) dr$$
$$+ \int_t^{\tau} (\pi'\sigma) \left(r, S^{\pi}_{t,s}(r), \pi(r) \right) dB(r) \ . \qquad (43)$$

With this change of variable, Proposition 9 states that u^* satisfies on $[0,T) \times (0,\infty)^d$ the equation:

$$\min \left\{ -\mathcal{G}u^* \ , \ \inf_{y \in \tilde{K}_1} \left(\delta(y) - \mathrm{diag}[s] Du^*(t,s) \right) \right\} \le 0$$

in the viscosity sense, where

$$\mathcal{G}u^*(t,s) := u^*_t(t,s) + \frac{1}{2} \left| \sigma'(t,s,\hat{\alpha}) \mathrm{diag}[s] Du^*(t,s) \right|^2$$
$$+ \frac{1}{2} \mathrm{Tr} \left[\mathrm{diag}[s] \sigma\sigma'(t,s,\hat{\alpha}) \mathrm{diag}[s] D^2 u^*(t,s) \right] \ ,$$
$$\text{and } \ \hat{\alpha} := \mathrm{diag}[s] Du^*(t,s) \ .$$

2. We argue by contradiction. Let $(t_0, s_0) \in [0,T) \times (0,\infty)^d$ and a C^2 $\varphi(t,s)$ be such that:

$$0 \ = \ (w^* - \varphi)(t_0, s_0) \ > \ (w^* - \varphi)(t,s) \text{ for all } (t,s) \neq (t_0, s_0) \ ,$$

and suppose that

$$\mathcal{G}\varphi(t,s) \ > \ 0 \text{ and } \mathrm{diag}[s_0] D\varphi(t_0, s_0) \ \in \ \mathrm{int}(K) \ .$$

Set $\hat{\pi}(t,s) := \mathrm{diag}[s] D\varphi(t,s)$. Let $0 < \alpha < T - t_0$ be an arbitrary scalar and define the neighborhood of (t_0, s_0):

$$\mathcal{N} := \left\{ (t,s) \in (t_0 - \alpha, t_0 + \alpha) \times s_0 e^{\alpha B} \ : \ \hat{\pi}(t,s) \in K \ \text{ and } \ -\mathcal{G}\varphi(t,s) \ge 0 \right\} \ ,$$

where B is again the unit closed ball of $\mathbb{R}^1$ and $s_0 e^{\alpha B}$ is the collection of all points $\xi \in \mathbb{R}^d$ such that $|\ln(\xi^i / s_0^i)| \le \gamma$. Since (t_0, s_0) is a strict maximizer of $(u^* - \varphi)$, observe that:

$$-4\beta := \max_{\partial \mathcal{N}} (u^* - \varphi) \ < \ 0 \ .$$

3. Let (t_1, s_1) be some element in $\mathcal{N}$ such that:

$$x_1 \ := \ u(t_1, s_1) \ge u^*(t_0, s_0) - \beta \ = \ \varphi(t_0, s_0) - \beta \ \ge \ \varphi(t_1, s_1) - 2\beta \ ,$$

and consider the controlled process

$$(\hat{S}, \hat{X}) \ := \ \left(S^{\hat{\pi}}_{t_1, s_1}, \ln W^{\hat{\pi}}_{t_1, s_1, w_1 e^{-\beta}} \right) \text{ with feedback control } \hat{\pi}(r) \ := \ \hat{\pi}\left(r, \hat{S}(r) \right) \ ,$$

which is well-defined at least up to the stopping time

$$\theta := \inf\left\{r > t_0 \;:\; \left(r, \hat{S}(r)\right) \not\in \mathcal{N}\right\} .$$

Observe that $u \leq u^* \leq \varphi - 3\beta$ on $\partial\mathcal{N}$ by continuity of the process $\hat{S}$. We then compute that:

$$\begin{aligned} x_1 - \beta - u\left(\theta, \hat{S}(\theta)\right) &\geq u(t_1, s_1) - 3\beta - \varphi\left(\theta, \hat{S}(\theta)\right) \\ &\geq \beta + \varphi(t_1, s_1) - \varphi\left(\theta, \hat{S}(\theta)\right) . \end{aligned} \tag{44}$$

4. From the definition of $\hat{\pi}$, the diffusion term of the difference $d\hat{X}(r) - d\varphi(r, \hat{S}(r))$ vanishes up to the stopping time θ. It then follows from (44), Itô's lemma, and (43) that

$$\begin{aligned} \hat{X}(\theta) - u\left(\theta, \hat{S}(\theta)\right) &\geq \beta + \int_{t_1}^{\theta} d\hat{X}(r) - d\varphi(r, \hat{S}(r)) \\ &= \beta + \int_{t_1}^{\theta} \mathcal{G}\varphi(r, \hat{S}(r)) \\ &\geq \beta \;>\; 0 \qquad \mathbb{P}-\text{a.s.} \end{aligned}$$

where the last inequality follows from the definition of the stopping time θ and the neighborhood $\mathcal{N}$. This proves that $W^{\hat{\pi}}_{t_1,s_1,w_1e^{-\beta}} > v(\theta, S^{\hat{\pi}}_{t_1,s_1}(\theta)$ $\mathbb{P}-$a.s., which is the required contradiction to DP2. □

7 Hedging under Gamma Constraints

In this section, we focus on an alternative constraint on the portfolio π. Let $Y :=$ diag$[S(t)]^{-1}\pi(t)W(t)$ denote the vector of number of shares of the risky assets held at each time. By definition of the portfolio strategy, the investor has to adjust his strategy at each time t, by passing the number of shares from $Y(t)$ to $Y(t+dt)$. His demand in risky assets at time t is then given by "$dY(t)$".

In an equilibrium model, the price process of the risky assets would be pushed upward for a large demand of the investor. We therefore study the hedging problem with constrained portfolio adjustment. This problem turns out to present serious mathematical difficulties. The analysis of this section is reported from [21]-[5], and provides a solution of the problem in a very specific situation. The extension to the multi-dimensional case is addressed in [6].

7.1 Problem Formulation

We consider a financial market which consists of one bank account, with constant price process $S^0(t) = 1$ for all $t \in [0, T]$, and one risky asset with price process evolving according to the Black-Scholes model:

$$S_{t,s}(u) := s\mathcal{E}\left(\sigma(B(t) - B(u)\right), \qquad t \leq u \leq T.$$

Here B is a standard Brownian motion in $I\!R$ defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We shall denote by $I\!F = \{\mathcal{F}(t), 0 \leq t \leq T\}$ the $\mathbb{P}$-augmentation of the filtration generated by B.

Observe that there is no loss of generality in taking S as a martingale, as one can always reduce the model to this case by judicious change of measure ($\mathbb{P}_0$ in the previous sections). On the other hand, the subsequent analysis can be easily extended to the case of a varying volatility coefficient.

We denote by $Y = \{Y(u), t \leq u \leq T\}$ the process of number of shares of risky asset S held by the agent during the time interval $[t, T]$. Then, by the self-financing condition, the wealth process induced by some initial capital w and portfolio strategy Y is given by:

$$W(u) \;=\; w + \int_t^u Y(r) dS_{t,s}(r), \qquad t \leq u \leq T.$$

In order to introduce constraints on the variations of the hedging portfolio Y, we restrict Y to the class of continuous semimartingales with respect to the filtration $I\!F$. Since $I\!F$ is the Brownian filtration, we define the controlled portfolio strategy $Y^{\alpha,\gamma}_{t,s,y}$ by:

$$Y^{\alpha,\gamma}_{t,y}(u) = y + \int_t^u \alpha(r) dr + + \int_t^u \gamma(r) \sigma dB(r), \quad t \leq u \leq T, \tag{45}$$

where $y \in I\!R$ is the initial portfolio and the *control* pair (α, γ) takes values in

$$\mathcal{B}_t := (\, L^\infty([t,T] \times \Omega \; ; \; \ell \otimes \mathbb{P})\,)^2 \; .$$

Hence a *trading strategy* is defined by the triple (y, α, γ) with $y \in I\!R$ and $(\alpha, \gamma) \in \mathcal{B}_t$. The associated wealth process, denoted by $W^{\alpha,\gamma}_{t,w,s,y}$, is given by:

$$W^{\alpha,\gamma}_{t,w,s,y}(u) = w + \int_t^u Y^{\alpha,\gamma}_{t,y}(r) dS_{t,s}(r), \qquad t \leq u \leq T \; . \tag{46}$$

We now formulate the Gamma constraint in the following way. Let Γ be a constant fixed throughout this section. Given some initial capital $w > 0$, we define the set of w-*admissible* trading strategies by:

$$\mathcal{A}_{t,s}(w) := \left\{ (y, \alpha, \gamma) \in I\!R \times \mathcal{B}_t \; : \; \gamma(.) \leq \Gamma \;\text{ and }\; W^{\alpha,\gamma}_{t,w,s,y}(.) \geq 0 \right\} \; .$$

As in the previous sections, We consider the super-replication problem of some European type contingent claim $g(S_{t,s}(T))$:

$$v(t,s) \;:=\; \inf \left\{ w \; : \; W^{\alpha,\gamma}_{t,w,s,y}(T) \geq g(S_{t,s}(T)) \;\text{ a.s. for some } (y, \alpha, \gamma) \in \mathcal{A}_{t,s}(w) \right\} \; . \tag{47}$$

7.2 The Main Result

Our goal is to derive the following explicit solution: $v(t, s)$ is the (unconstrained) Black-Scholes price of some convenient *face-lifted* contingent claim $\hat{g}(S_{t,s}(T))$, where the function $\hat{g}$ is defined by

$$\hat{g}(s) := h^{conc}(s) + \Gamma s \ln s \text{ with } h(s) := g(s) - \Gamma s \ln s ,$$

and h^{conc} denotes the concave envelope of h. Observe that this function can be computed easily. The reason for introducing this function is the following.

Lemma 3. *$\hat{g}$ is the smallest function satisfying the conditions*

$$\text{(i) } \hat{g} \geq g \text{ , and (ii) } s \longmapsto \hat{g}(s) - s \ln s \text{ is concave.}$$

The proof of this easy result is omitted.

Theorem 5. *Let g be a non-negative lower semi-continuous mapping on $I\!R_+$. Assume further that*

$$s \longmapsto \hat{g}(s) - C\, s \ln s \text{ is convex for some constant } C . \tag{48}$$

Then the value function (47) *is given by:*

$$v(t,s) = \mathbb{E}\left[\hat{g}\left(S_{t,s}(T)\right)\right] \text{ for all } (t,s) \in [0,T) \times (0,\infty) .$$

7.3 Discussion

1. We first make some comments on the model. Formally, we expect the optimal hedging portfolio to satisfy

$$\hat{Y}(u) = v_s(u, S_{t,s}(u)) ,$$

where v is the minimal super-replication cost; see Section 3.1. Assuming enough regularity, it follows from Itô's lemma that

$$d\hat{Y}(u) = A(u)du + \sigma(u, S_{t,s}(u))S_{t,s}(u)v_{ss}(u, S_{t,s}(u))dB(u) ,$$

where $A(u)$ is given in terms of derivatives of v. Compare this equation with (45) to conclude that the associated *gamma* is

$$\hat{\gamma}(u) = S_{t,s}(u)\, v_{ss}(u, S_{t,s}(u)) .$$

Therefore the bound on the process $\hat{\gamma}$ translates to a bound on sv_{ss}. Notice that, by changing the definition of the process γ in (45), we may bound v_{ss} instead of sv_{ss}. However, we choose to study sv_{ss} because it is a dimensionless quantity, i.e., if all the parameters in the problem are increased by the same factor, sv_{ss} still remains unchanged.

2. Observe that we only require an upper bound on the control γ. The similar problem with a lower bound on γ presents some specific difficulties, see [5] and [6]. In particular, the control $\int_0^t \alpha(r)dr$ has to be relaxed to the class of bounded variation processes...

3. The extension of the analysis of this section to the multi-asset framework is also a delicate issue. We send the interested reader to [6].

4. Intuitively, we expect to obtain a similar type solution to the case of portfolio constraints. If the Black-Scholes solution happens to satisfy the gamma constraint, then it solves the problem with gamma constraint. In this case v satisfies the PDE $-\mathcal{L}v = 0$. Since the Black-Scholes solution does not satisfy the gamma constraint, in general, we expect that function v solves the variational inequality:

$$\min\{-\mathcal{L}v, \Gamma - sv_{ss}\} = 0 . \tag{49}$$

5. An important feature of the log-normal Black and Sholes model is that the variational inequality (49) reduces to the Black-Scholes PDE $-\mathcal{L}v = 0$ as long as the terminal condition satisfies the gamma constraint (in a weak sense). From Lemma 3, the *face-lifted* payoff function $\hat{g}$ is precisely the minimal function above g which satisfies the gamma constraint (in a weak sense). This explains the nature of the solution reported in Theorem 5, namely $v(t,s)$ is the Black-Scholes price of the contingent claim $\hat{g}\left(S_{t,s}(T)\right)$.

6. One can easily check formally that the variational inequality (49) is the HJB equation associated to the stochastic control problem:

$$\tilde{v}(t,s) := \sup_{\nu \in \mathcal{N}} \mathbb{E}\left[g\left(S^{\nu}_{t,s}(T)\right) - \frac{1}{2}\int_t^T \nu(t)[S^{\nu}_{t,s}(r)]^2 dr\right] ,$$

where $\mathcal{N}$ is the set of all non-negative, bounded, and $I\!\!F-$ adapted processes, and:

$$S^{\nu}_{t,s}(u) := \mathcal{E}\left(\int_t^u [\sigma^2 + \nu(r)]^{1/2} dB(r)\right) , \quad \text{for } t \le u \le T .$$

The above stochastic control problem is a candidate for some dual formulation of the problem $v(t,s)$ defined in (47). Observe, however, that the dual variables ν are acting on the diffusion coefficient of the controlled process S^{ν}, so that the change of measure techniques of Section 3 do not help to prove the duality connection between v and $\tilde{v}$.

A direct proof of some duality connection between v and $\tilde{v}$ is again an open problem. In order to obtain the PDE characterization (49) of v, we shall make use of the dynamic programming principle stated directly on the initial formulation of the problem v.

7.4 Proof of Theorem 5

We shall denote

$$\hat{v}(t,s) := \mathbb{E}\left[\hat{g}\left(S_{t,s}(T)\right)\right] .$$

It is easy to check that $\hat{v}$ is a smooth function satisfying

$$\mathcal{L}\hat{v} \;=\; 0 \text{ and } s\hat{v}_{ss} \;\le\; \Gamma \text{ on } [0,T) \times (0,\infty) \;. \tag{50}$$

1. We start with the inequality $v \le \hat{v}$. For $t \le u \le T$, set

$$y \;:=\; \hat{v}_s(t,s) \;,\; \alpha(u) \;:=\; \mathcal{L}\hat{v}_s(u, S(u)) \;,\; \gamma(u) \;:=\; S_{t,s}(u)\hat{v}_{ss}(u, S(u)) \;,$$

and we claim that

$$(\alpha, \gamma) \;\in\; \mathcal{B}_t \text{ and } \gamma \le \Gamma \;. \tag{51}$$

Before verifying this claim, let us complete the proof of the required inequality. Since $g \le \hat{g}$, we have

$$\begin{aligned} g\left(S_{t,s}(T)\right) \le \hat{g}\left(S_{t,s}(T)\right) \;&=\; \hat{v}\left(T, S_{t,s}(T)\right) \\ &= \hat{v}(t,s) + \int_t^T \mathcal{L}\hat{v}(u, S_{t,s}(u))du + \hat{v}_s(u, S_{t,s}(u))dS_{t,s}(u) \\ &= \hat{v}(t,s) + \int_t^T Y_{t,y}^{\alpha,\gamma}(u)dS_{t,s}(u) \;; \end{aligned}$$

in the last step we applied Itô's formula to $\hat{v}_s$. Now, set $w := \hat{v}(t,s)$, and observe that $W_{t,w,s,y}^{\alpha,\gamma}(u) = \hat{v}(u, S_{t,s}(u)) \ge 0$ by non-negativity of the payoff function g. Hence $(y, \alpha, \gamma) \in \mathcal{A}_{t,s}(\hat{v}(t,s))$, and by the definition of the super-replication problem (47), we conclude that $v \le \hat{v}$.

It remains to prove (51). The upper bound on γ follows from (50). As for the lower bound, it is obtained as a direct consequence of Condition (48). Using again (50) and the smoothness of $\hat{v}$, we see that $0 = (\mathcal{L}\hat{v})_s = \mathcal{L}\hat{v}_s + \sigma^2 s\hat{v}_s s$, so that $\alpha = -\sigma^2\gamma$ is also bounded.

2. The proof of the reverse inequality $v \ge \hat{v}$ requires much more effort. The main step is the following dynamic programming principle which correspond to DP1 in Section 6.1.

Lemma 4. (Dynamic programming.) *Let $w \in I\!R$, $(y, \alpha, \gamma) \in \mathcal{A}_{t,s}(w)$ be such that $W_{t,w,s,y}^{\alpha,\gamma}(T) \ge g\left(S_{t,s}(T)\right)$ $\mathbb{P}-$a.s. Then*

$$W_{t,w,s,y}^{\alpha,\gamma}(\theta) \ge v\left(\theta, S_{t,s}(\theta)\right) \quad \mathbb{P} - a.s.$$

for all stopping time θ valued in $[t,T]$.

The obvious proof of this claim is similar to the first part of the proof of Lemma 2. We continue by by stating two lemmas whose proofs rely heavily on the above dynamic programming principle, and will be reported later. We denote as usual by v_* the lower semi-continuous envelope of v.

Lemma 5. *The function v_* is viscosity supersolution of the equation*

$$-\mathcal{L}v_* \;\ge\; 0 \text{ on } [0,T) \times (0,\infty) \;.$$

Lemma 6. *The function* $s \longmapsto v_*(t,s) - \Gamma s \ln s$ *is concave for all* $t \in [0,T]$.

Given the above results, we now proceed to the proof of the remaining inequality $v \geq \hat{v}$.

2.a. Given a trading strategy in $\mathcal{A}_{t,s}(w)$, the associated wealth process is a non-negative local martingale, and therefore a supermartingale. From this, one easily proves that $v_*(T,s) \geq g(s)$. By Lemma 6, $v_*(T,\cdot)$ also satisfies requirement (ii) of Lemma 3, and therefore

$$v_*(T,\cdot) \geq \hat{g} \; .$$

In view of Lemma 5, v_* is a viscosity supersolution of the equation $-\mathcal{L}v_* = 0$ and $v_*(T,\cdot) = \hat{g}$. Since $\hat{v}$ is a viscosity solution of the same equation, it follows from the classical comparison theorem that $v_* \geq \hat{v}$. □

Proof of Lemma 5 We split the argument in several steps.

1. We first show that the problem can be reduced to the case where the controls (α,γ) are uniformly bounded. For $\varepsilon \in (0,1]$, set

$$\mathcal{A}^{\varepsilon}_{t,s}(w) := \left\{ \, (y,\alpha,\gamma) \in \mathcal{A}_{t,s}(w) \; : \; |\alpha(.)| + |\gamma(.)| \leq \varepsilon^{-1} \, \right\},$$

and

$$v^{\varepsilon}(t,s) = \inf \Big\{ w \; : \; W^{\alpha,\gamma}_{t,w,s,y}(T) \geq g(S_{t,s}(T)) \, \mathbb{P} \\ - \text{a.s. for some } (y,\alpha,\gamma) \in \mathcal{A}^{\varepsilon}_{t,s}(w) \Big\} \; .$$

Let v^{ε}_* be the lower semi-continuous envelope of v^{ε}. It is clear that v^{ε} also satisfies the dynamic programming equation of Lemma 4.

Since

$$v_*(t,s) = \liminf{}_* \, v^{\varepsilon}(t,s) \; = \liminf_{\varepsilon \to 0, (t',s') \to (t,s)} v^{\varepsilon}_*(t',s') \; ,$$

we shall prove that

$$-\mathcal{L}v^{\varepsilon} \geq 0 \quad \text{in the viscosity sense,} \tag{52}$$

and the statement of the lemma follows from the classical stability result in the theory of viscosity solutions [7].

2. We now derive the implications of the dynamic programming principle of Lemma 4 applied to v^{ε}. Let $\varphi \in C^{\infty}(\mathbb{R}^2)$ and $(t_0,s_0) \in (0,T) \times (0,\infty)$ satisfy

$$0 \; = \; (v^{\varepsilon}_* - \varphi)(t_0,s_0) = \min_{(0,T)\times(0,\infty)} (v^{\varepsilon}_* - \varphi) \; ;$$

in particular, we have $v^{\varepsilon}_* \geq \varphi$. Choose a sequence $(t_n,s_n) \to (t_0,s_0)$ so that $v^{\varepsilon}(t_n,s_n)$ converges to $v^{\varepsilon}_*(t_0,s_0)$. For each n, by the definition of v^{ε} and the dynamic programming, there are $w_n \in [v^{\varepsilon}(t_n,s_n), v^{\varepsilon}(t_n,s_n) + 1/n]$, hedging strategies $(y_n,\alpha_n,\gamma_n) \in \mathcal{A}^{\varepsilon}_{t_n,s_n}(w_n)$ satisfying

$$W^{\alpha_n,\gamma_n}_{t_n,w_n,s_n,y_n}(\theta_n) - v^\varepsilon\left(t_n+t, S_{t_n,s_n}(t_n+t)\right) \geq 0$$

for every stopping time θ_n valued in $[t_n, T]$. Since $v^\varepsilon \geq v^\varepsilon_* \geq \varphi$,

$$w_n + \int_{t_n}^{\theta_n} Y^{\alpha_n,\gamma_n}_{t_n,y_n}(u) dS_{t_n,s_n}(u) - \varphi\left(\theta_n, S_{t_n,s_n}(\theta_n)\right) \geq 0 \ .$$

Observe that

$$\beta_n \ := \ w_n - \varphi(t_n, s_n) \longrightarrow 0 \ \text{ as } n \longrightarrow \infty \ .$$

By Itô's Lemma, this provides

$$M_n(\theta_n) \leq D_n(\theta_n) + \beta_n \ , \tag{53}$$

where

$$M_n(t) := \int_0^t \left[\varphi_s(t_n+u, S_{t_n,s_n}(t_n+u)) - Y^{\alpha_n,\gamma_n}_{t_n,y_n}(t_n+u)\right] dS_{t_n,s_n}(t_n+u)$$

$$D_n(t) := -\int_0^t \mathcal{L}\varphi(t_n+u, S_{t_n,s_n}(t_n+u))du \ .$$

We now chose conveniently the stopping time θ_n. For some sufficiently large positive constant λ and arbitrary $h > 0$, define the stopping time

$$\theta_n := (t_n+h) \wedge \inf\left\{u > t_n \ : \ \left|\ln\left(S_{t_n,s_n}(u)/s_n\right)\right| \geq \lambda\right\} \ .$$

3. By the smoothness of $\mathcal{L}\varphi$, the integrand in the definition of M_n is bounded up to the stopping time θ_n and therefore, taking expectation in (53) provides:

$$-\mathbb{E}\left[\int_0^{t\wedge\theta_n} \mathcal{L}\varphi(t_n+u, S_{t_n,s_n}(t_n+u))du\right] \geq -\beta_n \ ,$$

We now send n to infinity, divide by h and take the limit as $h \searrow 0$. The required result follows by dominated convergence. □

It remains to prove Lemma 6. The key-point is the following result. We refer the reader to [5] for the proof, and we observe that the argument in the original paper [21] is incomplete.

Lemma 7. *Let* $\left(\{a_n(u), \ u \geq 0\}\right)_n$ *and* $\left(\{b_n(u), \ u \geq 0\}\right)_n$ *be two sequences of real-valued, progressively measurable processes that are uniformly bounded in* n. *Let* (t_n, s_n) *be a sequence in* $[0,T) \times (0,\infty)$ *converging to* $(0, s)$ *for some* $s > 0$. *Suppose that*

$$\begin{aligned} M_n(t\wedge\tau_n) &:= \int_{t_n}^{t_n+t\wedge\tau_n} \left(z_n + \int_{t_n}^u a_n(r)dr + \int_{t_n}^u b_n(r)dS_{t_n,s_n}(r)\right) dS_{t_n,s_n}(u) \\ &\leq \ \beta_n + Ct\wedge\tau_n \end{aligned}$$

for some real numbers $(z_n)_n$*,* $(\beta_n)_n$*, and stopping times* $(\theta_n)_n \geq t_n$*. Assume further that, as* n *tends to zero,*

$$\beta_n \longrightarrow 0 \text{ and } t \wedge \theta_n \longrightarrow t \wedge \theta_0 \;\; \mathbb{P} - a.s.,$$

where θ_0 *is a strictly positive stopping time. Then:*
(i) $\lim_{n\to\infty} z_n = 0$.
(ii) $\lim_{t\searrow 0} \operatorname{ess\,inf}_{0\leq u\leq t} b(u) \leq 0$*, where* b *be a weak limit process of* $(b_n)_n$*.*

Proof of Lemma 6 We start exactly as in the previous proof by reducing the problem to the case of uniformly bounded controls, and writing the dynamic programming principle on the value function v^ε.

By a further application of Itô's lemma, we see that:

$$M_n(t) = \int_0^t \left(z_n + \int_0^u a_n(r)dr + \int_0^u b_n(r)dS_{t_n,s_n}(t_n+r) \right) dS_{t_n,s_n}(t_n+u)\ ,$$

where

$$\begin{aligned} z_n &:= \varphi_s(t_n, s_n) - y_n \\ a_n(r) &:= \mathcal{L}\varphi_s(t_n + r, S_{t_n,s_n}(t_n+r)) - \alpha_n(t_n+r) \\ b_n(r) &:= \varphi_{ss}(t_n + r, S_{t_n,s_n}(t_n+r)) - \frac{\gamma_n(t_n+r)}{S_{t_n,s_n}(t_n+r)}\ . \end{aligned}$$

Observe that the processes $a_n(. \wedge \theta_n)$ and $b_n(. \wedge \theta_n)$ are bounded uniformly in n since $\mathcal{L}\varphi_s$ and φ_{ss} are smooth functions. Also since $\mathcal{L}\varphi$ is bounded on the stochastic interval $[t_n, \theta_n]$, it follows from (53) that

$$M_n(\theta_n) \leq C\ t \wedge \theta_n \ +\ \beta_n$$

for some positive constant C. We now apply the results of Lemma 7 to the martingales M_n. The result is:

$$\lim_{n\to\infty} y_n \ =\ \varphi_s(t_0, y_0) \text{ and } \liminf_{n\to\infty,\ t\searrow 0} b(t) \ \leq\ 0.$$

where b is a weak limit of the sequence (b_n). Recalling that $\gamma_n(t) \leq \Gamma$, this provides that:

$$-s\varphi_{ss}(t_0, s_0) + \Gamma \geq 0\ .$$

Hence v_*^ε, and therefore v_*, is a viscosity supersolution of the equation $-s(v_*)_{ss} + \Gamma \geq 0$, and the required result follows by standard arguments in the theory of viscosity solutions. □

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